Exam 4b Calc 2 4/22/2011

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Give an example of a series which converges, but does not converge absolutely.

$\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ converges by AST, but $\sum_{n=0}^{\infty} \frac{1}{n}$ diverges because it is the harmonic series. Therefore $\sum_{n=0}^{\infty} \frac{(-1)^n}{n}$ does not converge absolutely

Great

2. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges or diverges.

$a_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$ is a p-series with $p = 2 > 1$

$\therefore \sum_{n=1}^{\infty} \frac{1}{n^2}$ converges

Great

3. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n+2}}$ converges or diverges.

$a_n = \frac{1}{\sqrt{n+2}}$ $\quad b_n = \frac{1}{n^{1/2}}$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges by p-series Test

$p = 1/2 \quad 1/2 \leq 1$

$p \leq 1$

$$\lim_{n\to\infty} \frac{\frac{1}{\sqrt{n+2}}}{\frac{1}{\sqrt{n}}} = \lim_{n\to\infty} \frac{1}{\sqrt{n+2}} \cdot \frac{\sqrt{n}}{1}$$

Nice!

$$\lim_{n\to\infty} \left(\frac{n}{n+2}\right)^{1/2} = \lim_{n\to\infty} \left(\frac{1}{1+\cancel{\frac{2}{n}}}\right)^{1/2}$$

$$\lim_{n\to\infty} = \left(\frac{1}{1}\right)^{1/2} = 1 = L$$

L is a positive, finite number so by the limit comparison test both series diverge

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$ converges or diverges.

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By the A.S.T.

1) $(-1)^n \rightarrow$ signs alternate. ✓

Excellent

2) $\lim_{n\to\infty} \left| \frac{(-1)^n}{\sqrt{n+2}} \right| = \lim_{n\to\infty} \frac{1}{\sqrt{n+2}} = 0.$ ✓

3) $f(x) = \frac{1}{\sqrt{n+2}} = \frac{1}{(n+2)^{\frac{1}{2}}} = (n+2)^{-\frac{1}{2}}$

$f'(x) = \underbrace{-\frac{1}{2}}_{<0} \underbrace{(n+2)^{-\frac{3}{2}}}_{>0} < 0.$ $\therefore$ f(x) is decreasing ✓

$\therefore$ the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n+2}}$ converges.

sin cos

π 0 — 0 ; −1 π −1 — 1 ; 0

5. Write the first 3 non-zero terms in the Taylor series for $f(x) = \cos x$ centered at $x = \pi$.

$a = \pi$

0 $f(x) = \cos x$ $f(\pi) = -1$

1 $f'(x) = -\sin x$ $f'(\pi) = 0$

2 $f''(x) = -\cos x$ $f''(\pi) = 1$

3 $f'''(x) = \sin x$ $f'''(x) = 0$

4 $f''''(x) = \cos x$ $f''''(x) = -1$

$$\cos x = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= \frac{-1}{0!}(x-\pi)^0 + \frac{1}{2!}(x-\pi)^2 - \frac{1}{4!}(x-\pi)^4$$

Well done

$$= -1 + \frac{1}{2}(x-\pi)^2 - \frac{1}{24}(x-\pi)^4$$

6. Find the radius of convergence for the power series $\sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$.

The $(-1)^n$ excluded because absolute value so will always be positive 1

$$\lim_{n\to\infty}\left|\frac{\frac{x^{2n+2}}{(2n+2)!}}{\frac{x^{2n}}{(2n)!}}\right| = \lim_{n\to\infty}\left|\frac{\overbrace{x^{2n+2}}^{x^{2n}(x)(x)}}{\underbrace{(2n+2)!}_{(2n+2)(2n+1)(2n!)}} \cdot \frac{(2n)!}{x^{2n}}\right|$$

$$\lim_{n\to\infty}\left|\frac{x^2}{(2n+2)(2n+1)}\right|$$

Excellent Job!

$$|x^2| \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)}$$

$$|x^2|(0) = 0$$

So the series converges absolutely by the Ratio test

and $R = \infty$ with $I = (-\infty, \infty)$

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod. Our Calc book is so unfair. There are these questions in it, like where you're supposed to explain why something is true or not, right? Which is stupid, because you know they can't put something like that on a multiple choice exam anyway, right? But our discussion section teacher said the professor was really excited about trying to see if we had conceptual knowledge, whatever that is, so we should pay special attention to this one that asked, like, if $\Sigma\, n^{-p}$ converges, then does $\Sigma\, n^{-p+0.0001}$ converge too. So it seems like 0.0001 isn't big enough to matter, but it feels like maybe a trick question, so I don't know."

Help Bunny by explaining what conclusions could be drawn here and why as clearly as possible.

Bunny, the thing is that there's a very fine line on which exponents make a series like that converge. $\Sigma \frac{1}{n^1}$ diverges, because it's the Harmonic Series, but $\Sigma \frac{1}{n^{1.0001}}$ converges, because it's a p-series with $p > 1$. So if you look at your problem with $p = -1.0001$, you have $\Sigma\, n^{-p}$ converging and $\Sigma\, n^{-p+0.0001}$ diverging. That definitely means you can't count on $\Sigma\, n^{-p+0.0001}$ converging.

8. Find the MacLaurin series for $f(x) = \arctan x$.

As long as $|x| < 1$

I remember: $\displaystyle\sum_{n=0}^{\infty} x^n \overset{*}{=} \frac{1}{1-x}$ because it's geometric

So substitute $(-x^2)$: $\displaystyle\sum_{n=0}^{\infty} (-x^2)^n \overset{*}{=} \frac{1}{1-(-x^2)}$

Rewrite: $\displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n} \overset{*}{=} \frac{1}{1+x^2}$

Antidifferentiate: $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} \overset{*}{=} \arctan x + C$

Finally note that $C = 0$ since $\arctan 0 = 0$, and we have

$$\boxed{\arctan x \overset{*}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}}$$

9. Determine whether the series $\displaystyle\sum_{n=2}^{\infty} \frac{3}{n \ln n}$ converges or diverges.

Integral Test

$$\int_2^{\infty} \frac{3}{x \ln x}\,dx = \lim_{b\to\infty} \int_{x=2}^{x=b} \frac{3}{x \cdot u} \cdot x\,du$$

Let $u = \ln x$

$\frac{du}{dx} = \frac{1}{x}$

$x\,du = dx$

$$= \lim_{b\to\infty} 3 \int_{x=2}^{x=b} \frac{1}{u}\,du$$

$$= 3 \lim_{b\to\infty} \ln|u| \Big|_{x=2}^{x=b}$$

$$= 3 \cdot \lim_{b\to\infty} \ln|\ln x| \Big|_2^b$$

$$= 3 \cdot \lim_{b\to\infty} \ln \ln b - \ln \ln 2$$

But as b grows without bound, $\ln b$ and thus $\ln \ln b$ grow without bound, so this integral diverges. Then by the Integral Test our series diverges too.

10. Find the interval of convergence for the series $\sum_{n=1}^{\infty} \frac{8^n x^n}{n+2}$.

Rat. Test!

$$\lim_{n\to\infty} \left| \frac{\frac{8^{n+1} x^{n+1}}{(n+1)+2}}{\frac{8^n x^n}{n+2}} \right| = \lim_{n\to\infty} \left| \frac{8^{n+1} \quad x^{n+1}}{n+3} \cdot \frac{n+2}{8^n x^n} \right|$$

$$= \lim_{n\to\infty} |8x| \cdot \frac{n+2}{n+3}$$

$$\overset{L'H}{=} \lim_{n\to\infty} |8x| \cdot \frac{1}{1}$$

$$= |8x| \quad \therefore \text{ It converges if } |8x| < 1,$$

or for $\frac{-1}{8} < x < \frac{1}{8}$

Now check endpoints:

If $x = \frac{1}{8}$ we have $\sum \frac{8^n (\frac{1}{8})^n}{n+2} = \sum \frac{1}{n+2}$

Limit Comparison to Harmonic Series (which diverges): $\lim_{n\to\infty} \frac{\frac{1}{n}}{\frac{1}{n+2}} = \lim_{n\to\infty} \frac{n+2}{n} = 1$

$\therefore$ The series diverges when $x = \frac{1}{8}$.

If $x = \frac{-1}{8}$ we have $\sum \frac{8^n(\frac{-1}{8})^n}{n+2} = \sum \frac{(-1)^n}{n+2}$

A.S.T. ✓ $(-1)^n$ makes sign alternate

✓ $\lim_{n\to\infty} \frac{1}{n+2} = 0$

✓ If $f(x) = \frac{1}{x+2}$, $f'(x) = \frac{0 \cdot (x+2) - 1 \cdot 1}{(x+2)^2} = \frac{-1}{(x+2)^2}$, which is a negative over a positive, so since the derivative is negative f is decreasing

So by A.S.T., if $x = -\frac{1}{8}$ the series converges

Therefore the interval of convergence is $\left[\frac{-1}{8}, \frac{1}{8}\right)$.