Examlet 1 **Foundations of Advanced Math** **2/4/11**

1. Show that the square of an odd integer is odd.

Well the definition of an odd integer is

if m is an odd integer then, $m = 2n+1$ where $n \in \mathbb{Z}$

thus if we square m...

$$(m)^2 = (2n+1)^2$$
$$= (2n+1)(2n+1)$$
$$= 4n^2 + 2n + 2n + 1$$
$$= 4n^2 + 4n + 1$$
$$= 2(2n^2 + 2n) + 1$$

where, by closure of the integers, we have that $2n^2 + 2n = p \in \mathbb{Z}$

So $2(p)+1$ is back to the defintion of what an odd integer is. Thus the square of <u>any</u> odd int. is odd. $\square$

Great.

2. a) There is no positive real number which is closest to 0.

Well suppose there is some positive real number m which is closest to 0.

But, we know that $1 > \frac{1}{2}$

So, if we multiply both sides by m $1m > \frac{1}{2}m$

it shows us that $\frac{m}{2}$ is closer to zero than m which contradicts our original statement that m is the closest real number to 0, proving that there is no real number closest to zero. Good

b) If x is an irrational, then x^2 is also irrational.

Well $\sqrt{2}$ is irrational because it cannot be written as an integer over an integer.

However $x^2 = (\sqrt{2})^2 = 2$, which is rational because it can be written as an integer over an integer $\frac{2}{1}$.

The proposition is false. Good

3. a) Show that an implication and its converse are logically equivalent.

Say we have any implication $P \Rightarrow Q$. It's converse is $Q \Rightarrow P$.

P	Q	$P \Rightarrow Q$	$Q \Rightarrow P$
T	T	T	T
T	F	F	T
F	T	T	F
F	F	T	T

Excellent!

Since the columns with ☆s are not identical in all cases, our implication is not logically equivalent with its converse. □

b) Determine whether $(P \wedge Q) \Rightarrow R$ is logically equivalent to $(P \Rightarrow R) \vee (Q \Rightarrow R)$

P	Q	R	$P \wedge Q$	$P \Rightarrow R$	$Q \Rightarrow R$	$(P \wedge Q) \Rightarrow R$	$(P \Rightarrow R) \vee (Q \Rightarrow R)$
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	T	T	T
T	F	F	F	F	T	T	T
F	T	T	F	T	T	T	T
F	T	F	F	T	F	T	T
F	F	T	F	T	T	T	T
F	F	F	F	T	T	T	T

Since the columns with ☆s are identical in all cases, the statements $(P \wedge Q) \Rightarrow R$ and $(P \Rightarrow R) \vee (Q \Rightarrow R)$ are logically equivalent. □

Nice!

4. Show that if $a \equiv_n 1$, then $a^2 \equiv_n 1$.

If $a \equiv_n 1$, then by definition

$n \mid (1-a)$, $n, a \in \mathbb{Z}$

or (by definition again) $n \cdot p = 1-a$, $p \in \mathbb{Z}$

Solving for a, we get ~~$a =$~~

$a = 1 - n \cdot p$

So, $a^2 = (1 - n \cdot p)^2$

$a^2 = n^2p^2 - 2np + 1$

$a^2 = n(np^2 - 2p) + 1$

$np^2 - 2p =$ some integer r, so

$a^2 = nr + 1$

Very nice!

$n(-r) = 1 - a^2$

$n \mid 1 - a^2$

$a^2 \equiv_n 1$

therefore if $a \equiv_n 1$, then $a^2 \equiv_n 1$.

5. The product of n odd integers is odd for any $n \geq 1$.

Let's use induction!

We will start w/ a base case, letting $n=1$.

The product of one odd integer is simply that specific odd integer, which can be written as $2p+1$, where $p \in \mathbb{Z}$.

Now let $n=2$ so that the product of 2 odd integers is equal to $(2p+1)(2p+1) = 4p^2+4p+1 = 2(2p^2+2p)+1$.

As we can see, the product is also odd.

Next, we will form an inductive hypothesis, and assume this is also true for when $n=k$.

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The product of k odd integers, which we assume is true is $(x_1 \cdot x_2 \cdot \ldots \cdot x_{k-1} \cdot x_k) = 2q+1$, where $q \in \mathbb{Z}$.

Nice!

Now let's see what happens when $n=k+1$.

The product of $k+1$ odd integers is $(x_1 \cdot x_2 \cdot \ldots \cdot x_{k-1} \cdot x_k \cdot x_{k+1})$.

We know that x_{k+1} is an odd integer, and we know from our inductive hypothesis that $(x_1 \cdot x_2 \cdot \ldots \cdot x_{k-1} \cdot x_k) = 2q+1$.

call it t, such that $t = 2d+1$, where $d \in \mathbb{Z}$.

Ultimately, we now simply have an odd integer times an odd integer:

$(2q+1)(2d+1) = 4dq + 2q + 2d + 1 = 2(2dq + q + d) + 1$,

and since that can be expressed as 2 times an integer plus one, we know the product of 2 odd integers is odd, and therefore we have proved the proposition to be true by mathematical induction.