Examlet 2 Foundations of Advanced Math 2/25/11

1. a) Let $A = \{1,2,4\}$ and $B = \{1,3,4\}$. What is $A \cup B$?

$A \cup B = \{1,2,3,4\}$ by def

because

def states that any x in $A \cup B$ is $\{x \mid x \in A \text{ or } x \in B\}$

b) Let $A = \{1,2,4\}$ and $B = \{1,3,4\}$. What is $A \cap B$?

$A \cap B = \{1,4\}$

for any element in $A \cap B$ is $\{x \mid x \in A \text{ and } x \in B\}$

c) Let $C = [0, 5]$ and $D = (4, 8)$. What is $C - D$?

$C - D = \{x \mid x \in C \text{ and } x \notin D\}$

$C - D = [0,4]$

Great

2. a) Let $\mathbb{N}^+ = \mathbb{N} - \{0\}$. Let $A_n = (0, n)$ for each $n \in \mathbb{N}^+$. What is $\bigcup_{n \in \mathbb{N}^+} A_n$?

$$\bigcup_{n \in \mathbb{N}^+} A_n = \mathbb{R}^+ = \{x \mid x \in \mathbb{R} \text{ and } x > 0\}$$

This is true since for any $x \in \mathbb{R}^+$ there exists $n \in \mathbb{N}^+$ with $n > x$, so $x \in A_n$, but for $x \leq 0$ we don't have $x \in (0, n)$ for any n.

b) Let I be a set such that for each $i \in I$, B_i is itself a set. Then $A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$.

If $x \in A \cap \bigcup_{i \in I} B_i$ then $x \in A \wedge x \in \bigcup_{i \in I} B_i$

this can be written $x \in A \wedge x \in B_i$ for some $i \in I$

Therefore we know $x \in A \cap B_i$ for some $i \in I$

which can be written $x \in \bigcup_{i \in I} (A \cap B_i)$

so $A \cap \bigcup_{i \in I} B_i \subseteq \bigcup_{i \in I} (A \cup B_i)$ since any element in the set on the left will also be in the set on the right.

If $x \in \bigcup_{i \in I} (A \cap B_i)$ then $x \in (A \cap B_i$ for some $i \in I$.

Therfore $x \in A \wedge x \in B_i$ for some $i \in I$

which can be written $x \in A \wedge x \in \bigcup_{i \in I} B_i$

Therfore we know $x \in A \cap \bigcup_{i \in I} B_i$

And since any element in $\bigcup_{i \in I} (A \cap B_i)$ is also in $A \cap \bigcup_{i \in I} B_i$ we know

$\bigcup_{i \in I} (A \cup B_i) \subseteq A \cap \bigcup_{i \in I} B_i$

Since they are subsets of each other, we know by def. of set equality that $A \cap \bigcup_{i \in I} B_i = \bigcup_{i \in I} (A \cap B_i)$ Excellent!

3. a) $\forall b \in \mathbb{R}$, if $b > 0$, then $0 < \frac{b}{2} < b$.

$b > 0 \Rightarrow \underbrace{\frac{b}{2} > \frac{0}{2}} \Rightarrow \underbrace{\frac{b}{2} > 0}$ (by multiplication principle)

So we know $\underbrace{0 < b}$ & $\underbrace{0 < \frac{b}{2}}$

And if we take $0 < \frac{b}{2}$ add $\frac{b}{2}$ to each side

$\underbrace{0 < \frac{b}{2}} \Rightarrow \underbrace{\frac{b}{2} + 0 < \frac{b}{2} + \frac{b}{2}} \Rightarrow \underbrace{\frac{b}{2} < b}$ (by addition principle)

We can now put these pieces together by the transitive principle to get

Great!

$0 < b$ & $0 < \frac{b}{2}$ & $\frac{b}{2} < b \Rightarrow \boxed{0 < \frac{b}{2} < b}$ giving us what we wanted to prove

b) $\forall b \in \mathbb{R}$, $|b| \geq 0$.

By defintion of absolute value we know...

if $b \geq 0$ $\Rightarrow |b| = b \Rightarrow |b| \geq 0$ (by substitution)

if $b < 0$ $\Rightarrow |b| = -b$

$-|b| = b$ (by multiplying each side by (-1))

Nice Job!

$-|b| < 0$ (by substituting $-|b| = b < 0$)

$|b| > 0$ (by multiplying each side by -1 & flipping the greater than/less than sign, this was proven to be valid in earlier proofs)

Thus we have $|b| \geq 0$ b/c since we know $|b| > 0$ it is not changing it at all to add the "or equal to" line as well. Thus since by the tricotomy princ we know b can only be $b=0$ $b<0$ $b>0$ thus we've proven $|b| \geq 0$ in all possible cases.

4. Let A be a set. Show that $A - A = \emptyset$.

$A - A = \{x \mid x \in A \wedge x \notin A\}$

If $x \in A - A$, then we know that $x \in A \wedge x \notin A$!

if $x \in A$ then ~~$\neg$~~ $x \notin A$ so no elements appear in both sets.

Because of this, we know that $A - A$ contains no elements.

Therefore any element $x \in A - A$ vacuously is also in $\emptyset$.

Therefore $A - A \subseteq \emptyset$.

Since $\emptyset$ contains no elements, we know that any $x \in \emptyset$ vacuously is contained in $A - A$.

Therefore $\emptyset \subseteq A - A$

Because $A - A \subseteq \emptyset$ and $\emptyset \subseteq A - A$, by def. of set equality we know that $A - A = \emptyset$.

Great!

5. a) $\forall a, b, c, d \in \mathbb{R}$, if $a > b$ and $c > d$, then $a + c > b + d$.

*Side note:
$b+c = c+b$
b/c we know addition to be communative

$a > b$ + $c > d$

$\downarrow$ $\downarrow$

$a+c > b+c$ $c+b > d+b$ (by addition principle)

Thus by transitivity we can say

$a+c > b+c > d+b \Rightarrow a+c > b+d$,

Excellent!

b) $\forall a, b, c, d \in \mathbb{R}$, if $\overset{3}{a} > \overset{0}{b}$ and $\overset{2}{c} > \overset{-4}{d}$, then $a - c > b - d$.

$3-2 > 0 + +4$

$1 > 4$

Consider the case where

$a = 3$ $b = 0$

$c = 2$ $d = -4$

Nice.

We see that $a > b \Rightarrow 3 > 0$ is valid

+ that $c > d \Rightarrow 2 > -4$ is also true.

But $a - c > b - d \Rightarrow 3 - 2 > 0 - (-4) \Rightarrow 1 > 4$

Which we know to be false b/c $1 \not> 4$.

Thus we have at least one contradiction and that proves the statement to be false.