Examlet 3 Foundations of Advanced Math 3/25/11

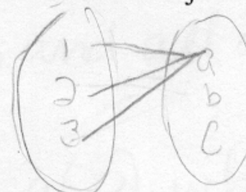
1. a) State the definition of an injection.

an injection is a function $f: A \to B$ such that

$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

Great

b) Give an example of a function from $A = \{1, 2, 3\}$ to $B = \{a, b, c\}$ which is not injective, and make it clear why it is not.



$f(x) = a$ is not injective.

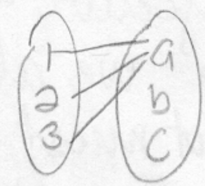
If it were, then $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

However $f(1) = a$ and $f(2) = a$ but $1 \neq 2$.

Therefore $f(x) = a$ from $A = \{1,2,3\}$ to $B = \{a,b,c\}$ is not an injective function

Excellent!

c) Give an example of a function from $A = \{1, 2, 3\}$ to $B = \{a, b, c\}$ which is not ~~injective~~ surjective, and make it clear why it is not.



$f(x) = a$ is not surjective either.

By definition, a surjective function $f: A \to B$ has $\forall b \in B$ some $a \in A$ such that $f(a) = b$.

Here, $f(1) = a$, $f(2) = a$ and $f(3) = a$. There are no elements of A such that $f(a) = b$ or $f(a) = c$.

Therefore $f(x) = a$ from $A = \{1,2,3\}$ to $B = \{a,b,c\}$ is not a surjective function

Nice

2. a) If there is an injection from $\mathbb{N}$ to a set A, then A is countable.

Counterexample: there is an injection from $\mathbb{N}$ to $\mathbb{R}$

$f: \mathbb{N} \to \mathbb{R} \quad f(x) = x$

but $\mathbb{R}$ is not countable

Great!

b) If there is an injection from a set A to $\mathbb{N}$, then A is countable.

$f: A \to \mathbb{N}$

Suppose there is an injection ~~$f: A \to \mathbb{N}$~~

Then let the image of f be B. $f: A \to B$ is a bijection, then, since f is by definition injective, and B only contains elements of the image of f.

Furthermore, B is a subset of $\mathbb{N}$. So there exists a bijection between A and a subset of $\mathbb{N}$.

Therefore A is countable.

Nice.

3. If $f: A \rightarrow B$ and $g: B \rightarrow C$ are surjective functions, then $g \circ f$ is surjective.

To show that $g \circ f$ is surjective we have to show that $\forall c \in C, \exists a \in A$ such that $(g \circ f)(a) = c$

We know that g is surjective so there exists some $b \in B$ such that $g(b) = c$. We also know that since f is surjective there exists an $a \in A$ such that $f(a) = b$. So,

$$(g \circ f)(a) = g(f(a)) \quad \text{We know } f(a) = b$$
$$= g(b) \quad \text{we know } g(b) = c$$
$$= c$$

Therefore $(g \circ f)(a) = c$ so for $\forall c \in C, \exists a \in A$ such that $(g \circ f)(a) = c$ and this proves that $g \circ f$ is surjective.

Beautiful!

4. a) If $f: A \to B$ has an inverse function g, then g has f as an inverse function also.

To say $f: A \to B$ has g as its inverse means $g: B \to A$, and that $g \circ f(a) = a$ and $f \circ g(b) = b$.

But all of that is exactly what we need to know to conclude that f is the inverse of g. □

b) Let $f: A \to B$ be a bijective function. Then there exists an inverse function g for f.

Since f is bijective, it must be surjective, so $\forall b \in B\ \exists a \in A$ such that $f(a) = b$. Then we define $g: B \to A$ by letting $g(b) = a$, and know this gives an image for every $a \in A$.

To confirm that g never sends a single $b \in B$ to two different $a_1, a_2 \in A$, note that this would mean $f(a_1) = b = f(a_2)$, but since f is injective $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$.

So we have created a function $g: B \to A$ so that $\forall a \in A$, $g \circ f(a) = a$ and $\forall b \in B$, $f \circ g(b) = b$, and thus g is an inverse for f, as desired. □

5. a) If $f: A \to B$ and $g: B \to C$ are functions and $g \circ f$ is injective, then f is injective.

Suppose $g \circ f$ is injective but f is not. If f is not injective, then $\exists\, a_1, a_2 \in A$ such that $f(a_1) = f(a_2)$ and $a_1 \neq a_2$. Since $f(a_1) = f(a_2)$, $g(f(a_1)) = g(f(a_2))$. However, $a_1 \neq a_2$, thus $g(f(a_1)) = g(f(a_2)) \not\Rightarrow a_1 = a_2$, contradiction our supposition that $g \circ f$ is injective. Since it is impossible for $g \circ f$ to be injective if f is not, f must be injective if $g \circ f$ is. □

Excellent!

b) If $f: A \to B$ and $g: B \to C$ are functions and $g \circ f$ is injective, then g is injective.

~~Suppose the $g \circ f$ is injective but g is not. If g is not injective then $\exists\, b_1, b_2 \in B$ such that $g(b_1) = g(b_2)$ and $b_1 \neq b_2$.~~

Take the functions $f: \mathbb{N} \to \mathbb{N}$ $\quad f(x) = x+1$

$g: \mathbb{N} \to \mathbb{N}$ $\quad x = 0 \to g(x) = 1$, $\; x \neq 0 \to g(x) = x$

In this case $g \circ f: \mathbb{N} \to \mathbb{N}$ $\quad x = 0 \to g(f(x)) = 1$, $\; x \neq 0 \to g(f(x)) = x+1$

$a_1 + 1 = a_2 + 1 \Rightarrow a_1 = a_2$ and $1 = x+1 \Rightarrow x = 0$, thus $g \circ f$ is injective. However, $g(0) = 1 = g(1)$ and $0 \neq 1$. Thus $g \circ f$ is injective and g is not, thus the statement is not true in all cases. □

Nice.