Examlet 4 Foundations of Advanced Math 4/16/10

1. a) State the definition of a transitive relation.

a relation $\sim$ on the set S is transitive iff $\forall a,b,c \in S$,
$\underline{a \sim b} \underline{\wedge} \underline{b \sim c} \underline{\Rightarrow} \underline{a \sim c}$

Great

b) Give an example of a relation on the set {1, 2, 3} which is symmetric but not reflexive.

$R = \{(1,2),(2,1)\}$

if the relation were reflexive, it would contain
$(1,1)$ <u>and</u>/or $(2,2)$.

Excellent!

2. a) Suppose that $\sim$ is an equivalence relation on the set $A = \{a, b, c, d, e\}$ and that $[a] = \{a, b\}$ and $[d] = \{d, e\}$. Write the partition $\mathcal{P}$ corresponding to $\sim$.

$\{\{a, b\}, \{c\}, \{d, e\}\}$

b) Suppose that $\mathcal{P}$ is the partition $\{\{1\}, \{2, 4\}, \{3, 5\}\}$ of the set $A = \{1, 2, 3, 4, 5\}$. Write the equivalence class of 2 under the corresponding relation.

$[2] = \{2, 4\}$ Great

3. Let R be a relation on $\mathbb{Z} \times \mathbb{Z}$ defined by $(a, b) \sim (c, d) \Leftrightarrow a - b = c - d$. Determine whether R is reflexive, symmetric, or transitive, and support your conclusions well.

REFLEXIVE

For R to be reflexive then $(a,b) \sim (a,b)$

So, $a-b = a-b$ and this is true $\therefore$ R is reflexive

SYMMETRIC

For R to be symmetric then $(a,b) \sim (c,d) \Rightarrow (c,d) \sim (a,b)$

So, $a-b = c-d$ which we know is true. Then we have that $c-d = a-b$ which we also know is true because it is the same as $a-b = c-d$ just written opposite. $\therefore$ R is symmetric

TRANSITIVE

For R to be transitive, $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f) \Rightarrow (a,b) \sim (e,f)$

So, we know $a-b = c-d$ and $c-d = e-f$. We can use do some substitution because $c-d = e-f$ and $a-b$, and we get that $a-b = e-f$ $\therefore$ R is transitive

R is an equuwallance relation because it is reflexive, symmetric and transitive

Excellent!

4. a) The sum of the degrees of the points in a graph is always even.

Let's induct on the number of edges in the graph.

If $n=1$, there are 2 vertices of degree 1 (and possibly others of degree 0) for a total degree of 2, which is even.

Now s'pose it's true when you have k edges, and consider a graph with $k+1$ edges. Adding the $k+1^{st}$ edge adds one to the degree of each vertex in that edge, so adds 2 to the even sum of the degrees of vertices in our graph with k edges, preserving and even sum. □

b) Suppose that a graph has n vertices. What is the largest number of them that can be of degree 3?

For n less than 4, it's 0 because there aren't enough other vertices.

For even $n \geq 4$, all n can be of degree 3: $n=4$

$n=6$

$n=8$

⋮

For odd $n \geq 5$, $n-1$ can be of degree 3. It can't be all n since that would violate part a, but we can add a vertex to the lower outside edge in the scheme above to attain $n-1$. □

5. a) If two relations R and S on A are reflexive, is $R \cup S$ reflexive?

$\forall a \in A$, since R is reflexive we have $(a,a) \in R$ and thus $(a,a) \in R \cup S$, so $R \cup S$ is reflexive. $\square$

b) If two relations R and S on A are transitive, is $R \cup S$ transitive?

No. For a counterexample consider $A = \{1, 2, 3\}$, with $R = \{(1,2)\}$ and $S = \{(2,3)\}$. We have R and S both vacuously transitive, but $(1,2) \in R \cup S$ and $(2,3) \in R \cup S$ even though $(1,3) \notin R \cup S$. $\square$