3.7 Uncountability

Lemma 1: $\mathbb{N} \times \mathbb{N}$ is countable.

Sketch of Proof: Well, we need a bijection from the naturals to the set of all ordered pairs of naturals. The diagram below suggests how this can be accomplished, with Exercise 10 of section 1.7 providing the most important of the details of the proof. The idea is that we imagine all of the ordered pairs laid out as shown, and then sweep through them in order along the indicated diagonals. Each ordered pair will be reached exactly once.

$$\begin{array}{ccccccc}
(0,0) & & (0,1) & & (0,2) & & \cdots \\
 & \swarrow & & \swarrow & & & \swarrow \\
(1,0) & & (1,1) & & (1,2) & & \cdots \\
 & \swarrow & & \swarrow & & & \swarrow \\
(2,0) & & (2,1) & & (2,2) & & \cdots \\
\vdots & \swarrow & \vdots & \swarrow & \vdots & & \ddots
\end{array}$$

□

Theorem 1: $\mathbb{Q}$ is countable.

Proof: Exercise 1.

Theorem 2 (Cantor): $\mathbb{R}$ is uncountable.

Proof: Well suppose, in hopes of obtaining a contradiction, that there were some $f\colon \mathbb{N} \to (0,1)$ which was a bijection. This suffices, by Exercise 7 in section 3.6, to show that there exists no bijection from $\mathbb{N}$ to $\mathbb{R}$. We will show that f is not in fact surjective by constructing an element $x \in \mathbb{R}$ which has no pre-image. We do this in steps. First, look at $f(0)$, and specifically look at the digit just to the right of the decimal point in its decimal expansion[*]. If that digit happens to be a 7, then we make the digit just to the right of the decimal place in x be a 3; if it's not a 7, then we make the corresponding digit in x be a 7. Next look at the digit one place over from the decimal point in $f(1)$, and similarly make that digit in x a 3 or 7 so that it's different from $f(1)$. Proceed with $f(2)$, $f(3)$, and so on. Then we have constructed an x which has no pre-image under f, since it is different from $f(0)$ in at least the digit immediately after the decimal point, different from $f(1)$ in at least the digit one place to the right of the decimal point, and so forth. This process disqualifies any purported bijection f, contradicting our supposition that such a bijection could exist. □

[*]This presupposes some facts about real numbers and their decimal expansions, specifically that decimal expansions are unique (enough) – the essential issue is that the same number can be written 0.7 or 0.6999999..., but that does not invalidate the reasoning here.

Exercises

1. Prove Theorem 1.

2. If there is an injection from $\mathbb{N}$ to a set A, then A is countable.

3. If there is an injection from a set A to $\mathbb{N}$, then A is countable.

4. If there is a surjection from a set B to $\mathbb{N}$, then B is countable.

5. If there is a surjection from $\mathbb{N}$ to a set B, then B is countable.
6. If A and B are countable sets, then $A \cup B$ is countable.
7. If A and B are disjoint denumerable sets, then $A \cup B$ is denumerable.
8. If A and B are countable sets, then $A \cap B$ is countable.
9. If A and B are denumerable sets, then $A \cap B$ is denumerable.
10. If A is a countable set, and $a \in A$, then $A - \{a\}$ is countable.
11. $\mathbb{Z} \times \mathbb{Z}$ is countable.
12. For any countable sets A and B, $A \times B$ is countable.
13. The irrationals are uncountable.
14. $\mathbb{R} \times \mathbb{R}$ is uncountable.
15. The union of two uncountable sets A and B is uncountable.
16. The union of a finite collection of finite sets is finite.
17. The union of any collection of finite sets is finite.
18. The union of a finite collection of countable sets is countable.
19. The union of a countable collection of countable sets is countable.
20. The union of any collection of countable sets is countable.
21. Any infinite set can be put in a bijective correspondence to one of its proper subsets.
22. If A is an infinite set, then $\mathcal{P}(A)$ is not equipollent to A.