5.3 The Peano Axioms I

You should have wondered at some point in your life exactly how many things needed to be accepted "on faith" as just obviously true in order to make the rest of the things commonly done in mathematics follow. The answer, technically, is none – mathematics proceeds in a hypothetical way, deducing the consequences *if* some collection of axioms holds. But for a more satisfying answer, this and the following sections provide an instance of developing a great deal of familiar material from a very short list of principles.

Definition: We call a set N a **Peano system** iff the following conditions are satisfied:

P1. $0 \in N$.
P2. For each $x \in N$, there is a unique element $x' \in N$ (we call x' the **successor** of x).
P3. $\forall x \in N, x' \neq 0$.
P4. $\forall x, y \in N, x' = y' \Rightarrow x = y$
P5. If $M \subseteq N$ for which $0 \in M$ and $\forall x \in M, x' \in M$, then $M = N$.

Definition: Given a Peano system N and $x, y \in N$, define their **sum** $x + y$ by

A1. $x + 0 = x$
A2. $x + (y') = (x + y)'$

Exercises

Prove the following statements, given that N is a Peano system.

1. $\forall x, y \in N, x + (y + 0) = (x + y) + 0$.

2. $\forall x, y, z \in N, x + (y + z) = (x + y) + z \Rightarrow x + (y + z') = (x + y) + z'$.

3. $\forall x, y, z \in N, x + (y + z) = (x + y) + z$.

4. $0 + 0 = 0$.

5. $\forall y \in N, 0 + y = y \Rightarrow 0 + y' = y'$.

6. $\forall y \in N, 0 + y = y$.

7. $\forall x \in N, x' + 0 = (x + 0)'$.

8. $\forall x, y \in N, x' + y = (x + y)' \Rightarrow x' + y' = (x + y')'$.

9. $\forall x, y \in N, x' + y = (x + y)'$.

10. $\forall y \in N, 0 + y = y + 0$.

11. $\forall x, y \in N, x + y = y + x \Rightarrow x' + y = y + x'$.

12. $\forall x, y \in N, x + y = y + x$.

5.4 The Peano Axioms II

The previous section developed the basic additive properties of the natural number system. This section extends that development to multiplication.

Definition: Given a Peano system N and $x, y \in N$, define their **product** $x \cdot y$ by

M1. $x \cdot 0 = 0$

M2. $x \cdot (y') = (x \cdot y) + x$

Exercises

Prove the following statements, given that N is a Peano system.

1. $\forall x, y \in N, x \cdot (y + 0) = x \cdot y + x \cdot 0.$

2. $\forall x, y, z \in N, x \cdot (y + z) = x \cdot y + x \cdot z \Rightarrow x \cdot (y + z') = x \cdot y + x \cdot z'.$

3. $\forall x, y, z \in N, x \cdot (y + z) = x \cdot y + x \cdot z.$

4. $\forall x, y \in N, x \cdot (y \cdot 0) = (x \cdot y) \cdot 0.$

5. $\forall x, y, z \in N, x \cdot (y \cdot z) = (x \cdot y) \cdot z \Rightarrow x \cdot (y \cdot z') = (x \cdot y) \cdot z'.$

6. $\forall x, y, z \in N, x \cdot (y \cdot z) = (x \cdot y) \cdot z.$

7. $0 \cdot 0 = 0.$

8. $\forall y \in N, 0 \cdot y = 0 \Rightarrow 0 \cdot y' = 0.$

9. $\forall y \in N, 0 \cdot y = 0.$

10. $\forall x \in N, x' \cdot 0 = 0.$

11. $\forall x, y \in N, x' \cdot y = x \cdot y + y \Rightarrow x' \cdot y' = x \cdot y' + y'.$

12. $\forall x, y \in N, x' \cdot y = x \cdot y + y.$

13. $\forall y \in N, 0 \cdot y = y \cdot 0.$

14. $\forall x, y \in N, x \cdot y = y \cdot x \Rightarrow x' \cdot y = y \cdot x'.$

15. $\forall x, y \in N, x \cdot y = y \cdot x.$

5.5 The Peano Axioms III

Here we include a few additional results that can be developed from the axioms already given. While it is impossible to list all the consequences of these axioms, these should dispel any impression that the results given previously are exhaustive.

Definition: We henceforth adopt the convention that $0' = 1$, $1' = 2$, $2' = 3$, $3' = 4$, and so forth.

Exercises

Prove the following statements, given that N is a Peano system.

1. $\forall y \in N$, with $y \neq 0$, $0 \neq 0 + y$.

2. $\forall x, y \in N$, with $y \neq 0$, $x \neq x + y \Rightarrow x' \neq x' + y$.

3. $\forall x, y \in N$, with $y \neq 0$, $x \neq x + y$.

4. $\forall y, z \in N$, $0 + y = 0 + z \Rightarrow y = z$.

5. $\forall x, y, z \in N$, $(x + y = x + z \Rightarrow y = z) \Rightarrow (x' + y = x' + z \Rightarrow y = z)$.

6. $\forall x, y, z \in N$, $x + y = x + z \Rightarrow y = z$.

7. $\forall y \in N$, with $y \neq 1$, $1 \neq 1 \cdot y$.

8. $\forall x, y \in N$, with $y \neq 1$, $x \neq x \cdot y \Rightarrow x' \neq x' \cdot y$.

9. $\forall x, y \in N$, with $y \neq 1$, $x \neq x \cdot y$.

10. $\forall y, z \in N$, $1 \cdot y = 1 \cdot z \Rightarrow y = z$.

11. $\forall x, y, z \in N$, $(x \cdot y = x \cdot z \Rightarrow y = z) \Rightarrow (x' \cdot y = x' \cdot z \Rightarrow y = z)$.

12. $\forall x, y, z \in N$, $x \cdot y = x \cdot z \Rightarrow y = z$.

5.6 The Peano Axioms IV

A sceptic might look at the previous sections and worry that, although they possess internal consistency, there is no particular reason to accept the axioms in their somewhat convoluted form as self-evident. Said differently, whether nine things or a thousand are taken on faith, we still might like to think mathematics is based on something other than hypotheticals. This section addresses that concern.

Definition: Given a set A, define $\boldsymbol{S(A)} = A \cup \{A\}$.

Definition: Let N be the collection of all sets N containing $\varnothing$ and such that $\forall x \in N$, $S(x) \in N$. Let $\mathbb{N}$ be the intersection of all sets in N.

Exercises

1. Write $S(\varnothing)$, $S(S(\varnothing))$, and $S(S(S(\varnothing)))$ explicitly. How many elements does each of these have?
2. Show that $\varnothing \in \mathbb{N}$.
3. Show that for each $x \in \mathbb{N}$, there is a unique element $\mathrm{S}(x) \in \mathbb{N}$.
4. Show that for any set A, $S(A) \neq \varnothing$.
5. Show that for any x, $\mathrm{y} \in \mathbb{N}$, $S(x) = S(y) \Rightarrow x = y$
6. Show that if $M \subseteq \mathbb{N}$ for which $\varnothing \in M$ and $\forall x \in M$, $S(x) \in M$, then $M = \mathbb{N}$.
7. Show that $\mathbb{N}$ is a Peano system.
8. Show that $1 + 1 = 2$.
9. Show that $2 + 2 = 4$.