Exam 2a   Calc 2   3/2/2012

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. For $\int_0^1 \sin\left(\sqrt{x}\right) dx$, the left-hand approximation using $n = 2$ subintervals is 0.3248 (to four decimal places). Find the midpoint and trapeziodal approximations with $n = 2$ subintervals.

$RHA = \frac{1}{2}\left(\sin\sqrt{1/2} + \sin\sqrt{1}\right) = .7456$ ← to 4 decimal, [illegible]

0 1/2 1   midpoint $= \frac{1}{2}\left(\sin\sqrt{1/4} + \sin\sqrt{3/4}\right) = \boxed{.62059}$

10   Trapezoidal is the average of right and left:

$$\frac{.7456 + .3248}{2} = \boxed{.5352.}$$

Excellent!

2. Evaluate $\int \sin^5 \theta \, d\theta$.

$\int \sin^4\theta \sin\theta \, d\theta$

$\int (\sin^2\theta)^2 \sin\theta \, d\theta$

$\int (1 - \cos^2\theta)^2 \sin\theta \, d\theta$

Let,

$u = \cos\theta$

$\frac{du}{d\theta} = -\sin\theta$

$d\theta = \frac{du}{-\sin\theta}$

$\int (1-u^2)^2 \sin\theta \frac{du}{-\sin\theta}$

$-\int (1 - 2u^2 + u^4)\, du$

$-\int du + 2\int u^2 \, du - \int u^4 \, du$

$-u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C$

$\boxed{-\cos\theta + \frac{2}{3}\cos^3\theta - \frac{1}{5}\cos^5\theta + C}$

Well done!

3. **Set up** an integral for the surface area obtained by rotating the curve $y = 1/x$ on [1,10] around the x-axis.

$$\int_1^{10} 2\pi \left(\frac{1}{x}\right)\sqrt{1+\left(-\frac{1}{x^2}\right)^2}\,dx$$

$$y = x^{-1}$$

$$dy = -\frac{1}{x^2}\,dx$$

$$= \boxed{2\pi \int_1^{10} \left(\frac{1}{x}\right)\sqrt{1+\frac{1}{x^4}}\,dx}$$

Good

4. Find the present value of an income stream of \$2000 per year, for a period of 10 years, if the continuous interest rate is 5%.

$$PV = \int_0^{M} P(t)\, e^{-rt}$$

$$PV = \int_0^{10} 2000\, e^{-.05t}\,dt$$

let $u = -.05t$

$$\frac{du}{-.05} = dt$$

$$PV = \frac{2000}{-.05}\int_0^{10} e^u\,dt$$

Great

$$= \frac{2000}{-.05}\, e^u \Big]_0^{10}$$

$$= \frac{2000}{-.05}\left(e^{-.05t}\Big]_0^{10}\right) = \frac{2000}{-.05}\left(e^{-.5} - e^0\right) = \frac{2000}{-.05}\left(e^{-.5} - 1\right) =$$

$$\boxed{\approx \$15{,}738.77}$$

5. Evaluate $\displaystyle\int_8^{\infty} \frac{dx}{\sqrt[3]{x}}$.

$$\lim_{b\to\infty} \int_8^b x^{-\frac{1}{3}}\,dx$$

$$\lim_{b\to\infty} \left(\frac{3x^{2/3}}{2}\right)\Bigg|_8^b$$

$$\lim_{b\to\infty} \left(\frac{3b^{2/3}}{2} - 6\right)$$

$$\boxed{\infty}$$

Good!

6. Bunny is a Calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, this is sooooo hard. I understand when they ask you to, like, work out a probability or something, right? But there was this problem on our test about why this one function wasn't a probability dense function, and that's totally unfair. How am I supposed to know it isn't one? The function was, like, $p(x) = \begin{cases} 0.2 & \text{for } 0 \le x \le 6 \\ 0 & \text{for } x<0 \text{ or } x>6 \end{cases}$."

   Explain clearly to Bunny how one can tell whether a function like this is (or is not) a p.d.f..

Well, Bunny, there are two rules that prove if something is a probability density function. The first is p(x) must be between 0 and 1 for all values of x (or equal to zero or one). This is true for the given function, since p(x) is equal to 0 or .2 for all x. So we're on the right track! The second rule of a p.d.f. is the integral of p(x) from negative infinity to positive infinity must equal 1. So:

$$\lim_{b\to\infty} \int_{-b}^{0} 0\,dx + \int_{0}^{6} 0.2\,dx + \lim_{b\to\infty} \int_{6}^{b} 0\,dx = 1$$

$$0 + 0.2x \Big|_0^6 + 0 = 1$$

$$0.2(6) - 0.2(0) = 1$$

$$1.2 \neq 1$$

Excellent!

So, since $\int_{-\infty}^{\infty} p(x)\,dx$ does not equal 1, the given function is not, in fact, a p.d.f.

7. Suppose the function $p(x) = \begin{cases} 0 & \text{for } x < 0 \\ 0.4e^{-0.4x} & \text{for } x \geq 0 \end{cases}$ is a probability distribution function for the probability that a shirt lasts x years before getting torn. Find the median number of years a shirt lasts.

For median,

$$\frac{1}{2} = \int_{-\infty}^{m} p(x)\, dx$$

$$\frac{1}{2} = \int_{-\infty}^{0} \cancel{p(x)}\, dx + \int_{0}^{m} p(x)\, dx$$

$$\frac{1}{2} = \int_{0}^{m} 0.4e^{-0.4x}\, dx$$

$$\frac{1}{2} = \frac{0.4}{-0.4}\left[e^{-0.4x}\right]_{0}^{m}$$

$$\frac{1}{2} = -\left[e^{-0.4m} - e^{-0.4(0)}\right]$$

$$\frac{1}{2} = -e^{-0.4m} + 1$$

$$e^{-0.4m} = 1 - \tfrac{1}{2} = \frac{1}{2}$$

$$-0.4m = \ln \tfrac{1}{2} \quad \text{[taking ln both sides]}$$

$$\boxed{m = -\frac{\ln 2}{0.4} = -1.733}$$

Excellent!

8. Show that $\int \sqrt{a^2 + x^2}\, dx$ can be transformed into $a^2 \int \sec^3 \theta \, d\theta$ by an appropriate substitution.

$= \int \sqrt{a^2 + x^2}\, dx$ let $x = a\tan\theta$

$dx = a\sec^2\theta\, d\theta$

$= \int \sqrt{a^2 + (a\tan\theta)^2} \cdot a\sec^2\theta\, d\theta$

$= \int \sqrt{a^2(1+\tan^2\theta)} \cdot a\sec^2\theta\, d\theta$

$= \int a\sqrt{1+\tan^2\theta} \cdot a\sec^2\theta\, d\theta$

$(\sqrt{\sec^2\theta})$

$= \int a^2 \sec\theta\, \sec^2\theta\, d\theta$

Great!

$= \int a^2 \sec^3\theta\, d\theta$ (a is a constant!)

$\boxed{= a^2 \int \sec^3\theta\, d\theta}$

$$\sin^2 x + \cos^2 x = 1$$
$$\tan^2 x + 1 = \sec^2 x$$

9. Derive the reduction formula $\int \sec^n x\,dx = \dfrac{\sec^{n-2} x \tan x}{n-1} + \dfrac{n-2}{n-1}\int \sec^{n-2} x\,dx$ (provided $n \neq 1$).

Parts!

$$\int \sec^n x\,dx = \int \sec^{n-2} x \cdot \sec^2 x\,dx \qquad u = \sec^{n-2} x \quad v = \tan x$$

$$= \sec^{n-2} x \cdot \tan x - \int (n-2)\cdot \sec^{n-2} x \cdot \tan^2 x\,dx \qquad u' = (n-2)\sec^{n-3} x \cdot \sec x \tan x \quad v' = \sec^2 x$$

$$= \sec^{n-2} x \cdot \tan x - (n-2)\int \sec^{n-2} x\,(\sec^2 x - 1)\,dx$$

$$= \sec^{n-2} x \tan x - (n-2)\int (\sec^n x - \sec^{n-2} x)\,dx$$

$$\int \sec^n x\,dx = \sec^{n-2} x \tan x - (n-2)\int \sec^n x\,dx + (n-2)\int \sec^{n-2} x\,dx$$

$$(n-1)\int \sec^n x\,dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x\,dx$$

$$\boxed{\int \sec^n x\,dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x\,dx}$$

10. Evaluate $\int \frac{1}{x\left(x^2+1\right)} dx$.

$$= \int \left(\frac{1}{x} + \frac{-x+0}{x^2+1} \right) dx$$

$$= \ln|x| - \frac{1}{2} \ln|x^2+1| + C$$

I wish:

$$\frac{1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$1 = A(x^2+1) + (Bx+C)x$$

If $x=0$:

$$1 = A \cdot 1 + 0B + 0C$$

$$\therefore A = 1$$

Matching x coefficients:

$$0 = C$$

Matching x^2 coefficients:

$$0 = A + B$$

$$0 = (1) + B$$

$$\therefore B = -1$$