Exam 3 Calc 2 4/6/2012

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Give an example of a geometric series which converges, and find its sum.

$1 + \frac{1}{5} + \frac{1}{25}$

$\frac{\frac{1}{5}}{1} = \frac{1}{5}$

$\frac{\frac{1}{25}}{\frac{1}{5}} = \frac{5}{25} = \frac{1}{5}$

$r = \frac{1}{5}$

$\sum_{n=0}^{\infty} \frac{1}{5^n}$ This will converge because $|r| = \frac{1}{5} < 1$

The sum equals $\frac{a}{1-r}$, and $a = \frac{1}{5^{(0)}} = 1$

so $\frac{1}{1-\frac{1}{5}} = \frac{1}{4/5} = \boxed{5/4}$

Excellent!

2. a) Write a Taylor polynomial of degree 4 centered at $x = 0$ for $f(x) = e^x$.

$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}$

Good

$\boxed{f(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24}}$

b) Write a Taylor polynomial of degree 4 centered at $x = 0$ for $f(x) = e^{3x}$.

$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = 1 + 3x + \frac{9x^2}{2!} + \frac{27x^3}{3!} + \frac{81x^4}{4!}$

$f(x) = \boxed{1 + 3x + \frac{9x^2}{2} + \frac{27x^3}{6} + \frac{81x^4}{24}}$

Great

3. Determine whether the series $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ converges or diverges.

Well, we know $\sum \frac{1}{2n}$ diverges because it is $\frac{1}{2}$ the Harmonic Series. Now, using the limit comparison test:

$$\lim_{n\to\infty} \left| \frac{\frac{1}{2n+1}}{\frac{1}{2n}} \right| = \lim_{n\to\infty} \frac{2n}{2n+1} \overset{L'H}{=} \lim_{n\to\infty} \frac{2}{2} = \underline{1}$$

Nice!

So since $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges, $\sum_{n=1}^{\infty} \frac{1}{2n+1}$ diverges also.

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ converges or diverges.

Try the Alternating Series Test:

✓ $(-1)^n$ makes it alternate

✓ $\lim_{n\to\infty} \frac{1}{2n+1} = 0$

✓ If $f(x) = \frac{1}{2x+1}$, $f'(x) = \frac{0(2x+1) - 1(2)}{(2x+1)^2} = \frac{-2}{(2x+1)^2}$,

which is negative over positive and therefore negative, so $f(x)$ is decreasing.

Thus by A.S.T., $\sum_{n=1}^{\infty} \frac{(-1)^n}{2n+1}$ converges.

5. Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{n!+2}$ converges or diverges.

Well, $0 < 2$,

so $n!+0 < n!+2$,

or $\frac{1}{n!+2} < \frac{1}{n!}$

Then since $\sum \frac{1}{n!}$ is convergent (from the Fake Quiz),
by the Comparison Test $\sum \frac{1}{n!+2}$ must be convergent.

6. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)}$ converges or diverges.

Using Integral Test

$$\int_2^{\infty} \frac{1}{n(\ln n)}\,dn$$

$$\lim_{b \to \infty} \int_2^{b} \frac{1}{n(\ln n)}\,dn$$

Let,

$u = \ln n$

$du = \frac{dn}{n}$

$n\,du = dn$

$$\lim_{b \to \infty} \int_{n=2}^{n=b} \frac{\cancel{n}\,du}{\cancel{n}\,u}$$

$$\lim_{b \to \infty} \ln|u| \Big|_{n=2}^{n=b}$$

$$\lim_{b \to \infty} \ln|\ln n| \Big|_2^b$$

$$\lim_{b \to \infty} \left(\ln|\ln b| - \ln|\ln 2| \right)$$

Here,

Since, $\ln|b|$ increases with no bounds so does $\ln|\ln b|$,

So the series diverges

Excellent!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Crap. They make all this Calc stuff so hard, but it's just because they want everyone to drop so they don't have so much to grade. My buddy from class told me about how they don't actually tell you some of the best tests for those series things, because you'd just use the same one every time instead of having to try different ones. It's like a conspiracy. So like when they say if the ratio test says 1 then you don't know, well my buddy showed me actually if you do the ratio test on $1/n^2$, you get 1, and you know $1/n^2$ converges, so obviously really ones where you get 1 from the ratio test converge, and they just won't admit it."

   Help Biff by explaining clearly whether his conclusions are valid or not, and why.

$\frac{1}{n^2}$ is a p-series, with $p > 1$ so it does converge, but we also know that if you try to use the ratio test on the harmonic series, you also get an L value of 1.

$\sum_{n=1}^{\infty} \frac{1}{n}$ $\quad \lim_{n\to\infty} \left| \frac{\frac{1}{n+1}}{\frac{1}{n}} \right| = \lim_{n\to\infty} \frac{1}{n+1} \cdot \frac{n}{1} = \lim_{n\to\infty} \frac{n}{n+1} \approx 1$

the harmonic series diverges so even though Biff found an example of a series with an L value = 1 by the ratio test that does converge, not all series that produce an L value of 1 do.

Excellent Answer.

8. Find the radius of convergence of the series $\sum_{k=0}^{\infty} \frac{(-1)^k (x-2)^k}{4^k}$.

Ratio Test

$$\lim_{n\to\infty} \left| \frac{\frac{(-1)^{k+1}(x-2)^{k+1}}{4^{k+1}}}{\frac{(-1)^k (x-2)^k}{4^k}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(-1)^k} \cdot -1^1 \; \cancel{(x-2)^k} \cdot (x-2)}{\cancel{4^k} \cdot 4} \cdot \frac{\cancel{4^k}}{\cancel{(-1)^k} \cancel{(x-2)^k}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{-(x-2)}{4} \right|$$

$$= \left| \frac{2-x}{4} \right| < 1$$

$\left| \frac{2-x}{4} \right| < 1$

$|2-x| < 4$

$-4 < 2-x < 4$

Radius of convergence $= 4$

$-6 < -x < 2$

$6 > x > -2$

$-2 < x < 6$

Well done.

9. Use a Taylor series with at least 3 nonzero terms to approximate $\int_0^{0.2} \frac{\sin x}{x}\, dx$.

$$\sin x \approx x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

$$\frac{\sin x}{x} \approx 1 - \frac{x^2}{3!} + \frac{x^4}{5!} = 1 - \frac{x^2}{6} + \frac{x^4}{120}$$

$$\int_0^{.2} \frac{\sin x}{x}\, dx \approx \int_0^{.2} \left(1 - \frac{x^2}{6} + \frac{x^4}{120}\right) dx$$

$$\approx x - \frac{x^3}{18} + \frac{x^5}{600} \Big|_0^{.2}$$

$$\approx .2 - \frac{.2^3}{18} + \frac{.2^5}{600}$$

Excellent!

10. Use a Taylor series with at least 4 nonzero terms to approximate $\ln\left(\frac{3}{2}\right)$.

Well, I know $\ln(1-x) \overset{*}{\approx} -\sum_{n=1} \frac{x^n}{n} = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4}$

so $\ln\left(1-\left(-\frac{1}{2}\right)\right) \overset{**}{\approx} -\left(-\frac{1}{2}\right) - \frac{\left(-\frac{1}{2}\right)^2}{2} - \frac{\left(-\frac{1}{2}\right)^3}{3} - \frac{\left(-\frac{1}{2}\right)^4}{4}$

$\approx \boxed{\frac{1}{2} - \frac{1}{8} + \frac{1}{24} - \frac{1}{64}} = \boxed{\frac{77}{192}} \approx \ln\left(\frac{3}{2}\right)$

* this is true when $|x| < 1$

** We can do this because $\left|-\frac{1}{2}\right| = \frac{1}{2} < 1$

so it <u>is</u> within the interval of convergence.

Nice!