Exam 1 Differential Equations 2/10/12

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. Check to see if $y(t) = 1 + t$ is a solution to the differential equation $\frac{dy}{dt} = \frac{y^2 - 1}{t^2 + 2t}$.

Well, does it work?

$y(t) = 1+t$

$\frac{dy}{dt} = 1$

$$1 = \frac{(1+t)^2 - 1}{t^2 + 2t} = \frac{1 + 2t + t^2 - 1}{t^2+2t} = \frac{2t+t^2}{t^2+2t}$$

$1 = 1$ ✓ it does!

So it is a solution

Wonderful!

2. Suppose that a certain differential equation has general solution $T(t) = 70 + c\, e^{-0.2t}$. Find a particular solution satisfying the initial condition $T(0) = 110$.

$T(t) = 70 + ce^{-.2t}$

$110 = 70 + Ce^{-.2(0)} = 70 + C$

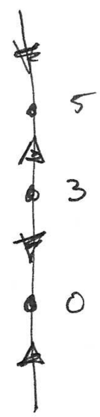
$C = 40$

$T(t) = 70 + 40e^{-0.2t}$

Great.

3. Sketch a phase line for the differential equation $\frac{dP}{dt} = -0.003P(3-P)(5-P)$.

EQB $P=0$, $P=3$, $P=5$



Great!

ans

$P=-1$ $-$ $-$ $+$ $+$

$P' = -0.003(-1)(3-(-1))(5-(-1)) > 0$

$P=1$ $-$ $+$ $+$ $+$

$P' = -0.003(1)(3-1)(5-1) < 0$

$P=4$ $-$ $-$ $+$

$P' = -0.003(4)(3-4)(5-4) > 0$

$P=6$ $-$ $-$ $-$

$P' = -0.003(6)(3-6)(5-6) < 0$

4. Find a general solution to the differential equation $\frac{dB}{dt} = 0.05B - 1000$. For full credit your solution should give B explicitly as a function of t.

$\frac{dB}{dt} = 0.05(B-20{,}000)$

$(B-20{,}000)^{-1}\,dB = 0.05\,dt$

$\int (B-20{,}000)^{-1}\,dB = 0.05t + C$

$\ln|B-20{,}000| = 0.05t + C$

$|B-20{,}000| = Ae^{0.05t}$

$\boxed{B = Ae^{0.05t} + 20{,}000}$

$\int (B-20{,}000)^{-1}$

Wonderful.

$u = B-20{,}000$

$du = 1\,dB$

$\ln|B-20{,}000|$

Abs should be taken care of by A.

5. Consider the differential equation $\frac{dB}{dt} = 0.05B - 1000$. Use Euler's method with step size $\Delta t = 5$ to approximate $B(10)$, given that $B(0) = 26000$.

t	B	$\frac{dB}{dt}$	ΔB
0	26000	300	1500
5	27500	375	1875
10	29375		

$\frac{\Delta B}{\Delta t} = 300, \quad \Delta B = 300\Delta t$

$0.05(26000) - 1000 = 300$

$0.05(27500) - 1000 = 375$

$\frac{\Delta B}{\Delta t} = 375, \quad \Delta B = 375\Delta t$

Good!

$\boxed{B(10) = 29375}$

6. Find the power series expansion for the general solution up to degree four to the differential equation $\frac{dy}{dt} = 2y$.

$$y = a_0 + a_1 t + a_2 t^2 + a_3 t^3 + a_4 t^4 + a_5 t^5$$
$$y' = a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4$$

$$a_1 + 2a_2 t + 3a_3 t^2 + 4a_4 t^3 + 5a_5 t^4 = 2a_0 + 2a_1 t + 2a_2 t^2 + 2a_3 t^3 + 2a_4 t^4 + 2a$$

$a_1 = 2a_0$	$a_1 = 2a_0$	$a_1 = 2a_0$
$2a_2 = 2a_1$	$a_2 = a_1$	$a_2 = 2a_0$
$3a_3 = 2a_2$	$a_3 = \frac{2}{3}a_2$	$a_3 = \frac{4}{3}a_0$
$4a_4 = 2a_3$	$a_4 = \frac{1}{2}a_3$	$a_4 = \frac{2}{3}a_0$

Excellent!

$$y = a_0 + 2a_0 t + 2a_0 t^2 + \frac{4}{3}a_0 t^3 + \frac{2}{3}a_0 t^4$$

7. Biff is a differential equations student at Enormous State University. Biff says "Man, I was pretty mixed up in D.E. until we learned about phase lines. That's my kinda stuff, 'cause you can just solve a little equation, plug some numbers in, draw some cool little arrows, and bam, there you go. I love it! I say screw all that other stuff they say you have to learn, just do it all with phase lines!"

   Explain clearly to Biff which situations phase lines are appropriate for, and something about limitations of this approach.

Phase lines are acceptable only in the case of autonomous differential equations — in other words, equations where the rate of change depends only on the dependent variable. These don't change with time, so they are the only place we can get stable equilibria. Without this, you would need a new phase line for every value of t, because the equilibrium values would be constantly changing, so a single phase line would be next to worthess.

Even apart from that, phase lines only describe the general behavior based on various initial conditions. If you are looking for a specific answer instead of the long term behavior, the phase line would not help you.

Excellent!

8. Find a general solution to the differential equation $\dfrac{dy}{dt} = \dfrac{3}{t}y + t^5$.

This equation is linear so,

$\frac{dy}{dt} - \frac{3}{t}y = t^5$ $\qquad \mu(t) = e^{\int -\frac{3}{t}dt} = e^{-3\ln t} = e^{\ln t^{-3}} = t^{-3} = \frac{1}{t^3}$

$\frac{1}{t^3}\frac{dy}{dt} - \frac{1}{t^3}\frac{3}{t}y = \frac{1}{t^3}t^5$

$\frac{d}{dt}\left(\frac{1}{t^3}y\right) = t^2$

Integrating you get,

$\frac{y}{t^3} = \frac{t^3}{3} + c$

Nice Job

$y = \frac{(t^3)(t^3)}{3} + (t^3)c$

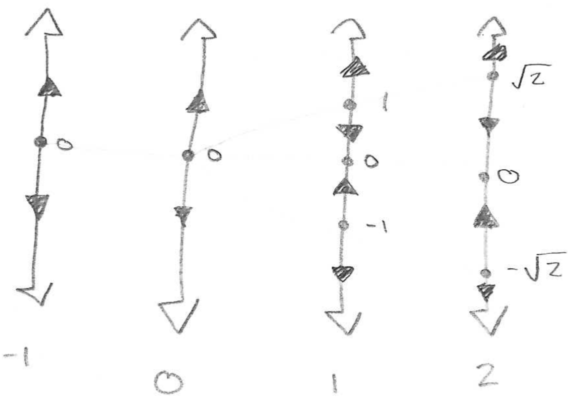
$\boxed{y(t) = \frac{t^6}{3} + t^3 c}$

9. **Sketch the bifurcation diagram** for the differential equation $\frac{dy}{dt} = y^3 - \alpha y$. Include direction arrows on the phase lines and make clear the exact α values where bifurcations occur.

$\alpha = 0$. eq pt $\rightarrow 0$

$\alpha = 1$ eq pt $\rightarrow -1, 0, 1$

$\alpha = 2$ eq pt $\rightarrow -\sqrt{2}, 0, \sqrt{2}$



$\alpha = 0$ is a bifurcation pt

Excellent!

10. A pond containing 100,000 liters of water is located near a metal-plating facility. The factory begins leaking chromium into the stream feeding the pond so that each minute 50 liters of water containing 0.000002 kg of chromium per liter flows into the pond. Well-mixed water is flowing out of the pond at 20 liters per minute. Write a differential equation for the amount of chromium in the pond at time *t*, find a general solution, and find a particular solution satisfying the condition that the pond began with no chromium. [Special thanks and 50% of all profits generated go to Marty St. Clair for providing semi-plausible background!]

Differential Equation!

$$\frac{dP}{dt} = \underbrace{50}_{\text{Amount}} \cdot \underbrace{0.000002}_{\text{Concentration}} - \underbrace{20}_{\text{Amount}} \underbrace{\frac{P}{100{,}000 + 30t}}_{\text{Concentration}}$$

$$\mu(t) = e^{20\int \frac{1}{100{,}000+30t}}$$

$u = 100{,}000 + 30t$

$du = 30\,dt$

General Solution!

$$\frac{dP}{dt} + \frac{20}{100{,}000 + 30t} P = 10^{-4}$$

Look! It's linear!

$$\mu(t) = e^{\frac{20}{30}\int u^{-1}\,du}$$

$$= e^{\frac{2}{3}\ln|100{,}000 + 30t|}$$

Note the $2/3$ take care of abs.

$$= (100{,}000 + 30t)^{2/3}$$

$$\frac{dP}{dt}(100{,}000 + 30t)^{2/3} + 20(100{,}000 + 30t)^{-1/3} P = 10^{-4}(100{,}000 + 30t)^{2/3}$$

$$\frac{d}{dt}\left(P(10^5 + 30t)^{2/3}\right) = 10^{-4}(10^5 + 30t)^{2/3}$$

$u = 10^5 + 30t$

$du = 30\,dt$

$$P(10^5 + 30t)^{2/3} = 10^{-4}\int (10^5 + 30t)^{2/3}\,dt$$

$$= \frac{10^{-4}}{30}\int u^{2/3}\,du$$

$$= \frac{10^{-4}}{30}\frac{3}{5}(10^5 + 30t)^{5/3} + C$$

$$P = \frac{10^{-4}}{50}(10^5 + 30t) + C(10^5 + 30t)^{-2/3}$$

$$\boxed{P = 10^{-5}\frac{1}{5}(10^5 + 30t) + C(10^5 + 30t)^{-2/3}}$$

Particular Solution!

$$P(0) = 0 = 10^{-5}\frac{1}{5}(10^5) + C(10^5)^{-2/3}$$

$$0 = \frac{1}{5} + C\,10^{-\frac{10}{3}}$$

$$-\frac{1}{5} = C\,10^{-\frac{10}{3}}$$

$$C = -\frac{1}{5}10^{\frac{10}{3}}$$

$$\boxed{P(t) = 10^{-5}\frac{1}{5}(10^5 + 30t) - \frac{1}{5}10^{\frac{10}{3}}(10^5 + 30t)^{-2/3}}$$