Exam 2a Differential Equations 3/16/12

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. Find a solution to the system

$$\frac{dx}{dt} = 2x$$
$$\frac{dy}{dt} = 3y$$

Guess: $\underline{x(t) = k_1 e^{2t}}$ $\underline{y(t) = k_2 e^{3t}}$

Check: $\frac{dx}{dt} = 2k_1 e^{2t}$ where $k_1 e^{2t} = x$ so $\frac{dx}{dt} = 2x$

$\frac{dy}{dt} = 3k_2 e^{3t}$ where $k_2 e^{3t} = y$ so $\frac{dy}{dt} = 3y$

Excellent!

works!

2. Give an example of a partially decoupled system of differential equations.

$\frac{dx}{dt} = xy$

$\frac{dy}{dt} = y$

$\frac{dx}{dt}$ is dependent upon both x and y

where $\frac{dy}{dt}$ is only dependent upon y so it is partially decoupled system

Excellent!

3. Find all equilibrium points of the system

$$\frac{dR}{dt} = 2R - 1.2RF$$

$$\frac{dF}{dt} = -F + 0.9RF = F(-1 + .9R)$$

$\frac{dF}{dt} = 0$, so $0 = F(-1 + 0.9R)$ so either $F = 0$ or $R = 10/9$

if $F = 0$, then $0 = 2R - 1.2(0)(R)$

$0 = 2R$, so $R = 0$

if $R = 10/9$, then $0 = 2(10/9) - 1.2(10/9)F$

$\frac{20}{9} = \frac{4}{3}F$

$5/3 = F$

Wonderful!

Equilibria

$(0, 0)$

$(10/9, 5/3)$

4. Suppose that you know $x(t) = k_2e^{2t} - \frac{k_1}{3}e^{-t}$, $y(t) = k_1e^{-t}$ is a general solution to a system of differential equations. Find the solution satisfying the initial condition $\mathbf{Y}(0) = (x(0), y(0)) = (2, 1)$.

$x(0) = k_2e^{2(0)} - \frac{k_1}{3}e^{0} = 2$ and $y(0) = k_1e^{0} = 1$

$k_1 = 1$

$k_2 - \frac{k_1}{3} = 2$

$k_2 = 2 + \frac{k_1}{3}$ and $k_1 = 1$

So, $k_2 = 2 + \frac{1}{3}$

$k_2 = 7/3$

Well done!

So,

$\boxed{x(t) = 7/3e^{2t} - \frac{1}{3}e^{-t} \text{ and } y(t) = e^{-t}}$

5. Consider the system $$\frac{dR}{dt} = 2R - 1.2RF$$ $$\frac{dF}{dt} = -F + 0.9RF$$. Let $R(0) = 2$ and $F(0) = 1.5$, and use Euler's method with step size $\Delta t = 2$ to approximate $R(2)$ and $F(2)$.

t	$\frac{dR}{dt}$	R	ΔR	$\frac{dF}{dt}$	F	ΔF
0	.4	2	.8	1.2	1.5	2.4
2		2.8			3.9	

so $R(2) = 2.8$ and $F(2) = 3.9$ with Euler's method ~~+~~ a step size $\Delta t = 2$

Great!

6. Find a solution to the system $\begin{aligned} \frac{dx}{dt} &= 3x+2y \\ \frac{dy}{dt} &= 5x \end{aligned}$.

$\Longrightarrow \quad x'' = 3x' + 2y'$

Substituting $y' = 5x$:

$$x'' = 3x' + 2(5x)$$

or $\quad x'' - 3x' - 10x = 0$

Prob'ly $\quad x(t) = e^{st}$

$$x'(t) = se^{st}$$

$$x''(t) = s^2 e^{st}$$

So $\quad s^2 e^{st} - 3se^{st} - 10e^{st} = 0$

$$e^{st}(s^2 - 3s - 10) = 0$$

$$e^{st}(s-5)(s+2) = 0$$

$$s = 5 \text{ or } s = -2$$

Thus $\quad x(t) = k_1 e^{5t} + k_2 e^{-2t}$

And $\quad y'(t) = 5x(t)$

$$= 5(k_1 e^{5t} + k_2 e^{-2t})$$

Hence $\quad y(t) = k_1 e^{5t} - \frac{5k_2}{2} e^{-2t}$

7. Let $y(t) = 5$. Find $\mathcal{L}[y]$, and note any necessary restrictions.

$$\mathcal{L}[y(t)] = \int_0^\infty y(t)e^{-st}\,dt$$

$$\mathcal{L}[5] = \lim_{b\to\infty} \int_0^b 5e^{-st}\,dt$$

$$= \lim_{b\to\infty} 5\int_0^b e^{-st}\,dt$$

$$= \lim_{b\to\infty} 5\left[\frac{-1}{s}e^{-st}\right]_0^b$$

$$= \lim_{b\to\infty} 5\left[\left(\frac{-1}{s}e^{-s(b)}\right) - \left(\frac{-1}{s}e^{-s(0)}\right)\right]$$

$$= 5\left[(0) - \left(\frac{-1}{s}\right)\right]$$

$$= \frac{5}{s} \quad \text{when } s > 0$$

well done!

8. Compute the inverse Laplace transform $\mathcal{L}^{-1}\left[\frac{5}{(s-1)(s-2)}\right]$.

I wish

$$\frac{5}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$$

$$5 = A(s-2) + B(s-1)$$

<u>If $s=1$:</u>

$5 = -A$

$\therefore A = -5$

<u>If $s=2$:</u>

$5 = B$

So $$\mathcal{L}^{-1}\left[\frac{5}{(s-1)(s-2)}\right] = \mathcal{L}^{-1}\left[\frac{-5}{s-1} + \frac{5}{s-2}\right]$$

$$= \mathcal{L}^{-1}\left[\frac{-5}{s-1}\right] + \mathcal{L}^{-1}\left[\frac{5}{s-2}\right]$$

$$= -5 \cdot \mathcal{L}^{-1}\left[\frac{1}{s-1}\right] + 5 \cdot \mathcal{L}^{-1}\left[\frac{1}{s-2}\right]$$

$$= -5 \cdot e^{1t} + 5 \cdot e^{2t}$$

9. Suppose $a \geq 0$. Compute the Laplace transform of the function

$$r_a(t) = \begin{cases} 0 & \text{if } t < a \\ k(t-a) & \text{if } t \geq a \end{cases}.$$

$r_a(t) = kt\mu_a(t) - ka\mu_a(t)$ where $\mu_a(t) = \begin{cases} 0 & \text{if } t < a \\ 1 & \text{if } t \geq a \end{cases}$

Well $\mathcal{L}(-ka\mu_a(t)) = \dfrac{-ka}{se^{sa}}$.

Clever, though not helpful!

So we need to compute $\mathcal{L}(kt\mu_a(t)) = k\int_0^a 0\,dt + k\int_a^\infty te^{-st}\,dt$

$= k\int_a^\infty te^{-st}\,dt$

$u = t \quad dv = e^{-st}$

$du = 1 \quad v = \frac{1}{-s}e^{-st}$

$= \lim_{b\to\infty}\left[kt \cdot \frac{1}{-s}e^{-st}\right]_a^b - k\int_a^\infty \frac{1}{-s}e^{-st}\,dt$

$= \frac{ka}{s}e^{-sa} - k\lim_{b\to\infty}\left[\frac{1}{s^2}e^{-st}\right]_a^b$

$= \dfrac{ka}{se^{sa}} + k\frac{1}{s^2}e^{-sa}$

Thus $\mathcal{L}(r_a(t)) = \dfrac{ka}{se^{sa}} + \dfrac{k}{s^2}e^{-sa} - \dfrac{k}{se^{sa}}$ Nice.

$= \dfrac{k}{s^2e^{sa}}$ for $s > 0$

10. Consider the second-order differential equation $y'' + \beta y' + 12y = 0$.

a) Let $\beta = 8$. Find a solution to the equation.

$y'' + 8y' + 12y = 0$

Prob'ly $y = e^{st}$

so $y' = s\cdot e^{st}$

$y'' = s^2 \cdot e^{st}$

$(s^2 e^{st}) + 8(s\cdot e^{st}) + 12(e^{st}) = 0$

$e^{st}(s^2 + 8s + 12) = 0$

$e^{st}(s+6)(s+2) = 0$

So $s = -6$ or $s = -2$

Then our general solution is $y = k_1 e^{-6t} + k_2 e^{-2t}$

b) Find a value for the parameter β for which $y(t) = e^{-3t}$ is a solution to this equation.

Well, if $y = e^{-3t}$

then $y' = -3e^{-3t}$

$y'' = 9e^{-3t}$

So for this to be a solution, it must satisfy:

$y'' + \beta y' + 12y = 0$

$(9e^{-3t}) + \beta(-3e^{-3t}) + 12(e^{-3t}) = 0$

$e^{-3t}(9 - 3\beta + 12) = 0$

$21 = 3\beta$

$\therefore \beta = 7$