2 Sets

2.1 Introduction to Sets

A set is an object S for which, given an object x, we can determine whether x is or is not an element of S. We will frequently represent the situation where x is an element of S by $x \in S$, and the alternative situation where x is not an element of S by $x \notin S$. You might notice that it is customary to use capital letters for the names of sets, and lower case letters for elements, although this is not a strict rule – in fact it will be very useful to consider sets which themselves contain other sets.

We will write $A = \{1, 2, 3\}$, for instance, when we mean the set which contains the element 1, the element 2, and the element 3, but no other elements. Notice the set braces "{" and "}" used here. Don't mix them up with parentheses or brackets, which will be used for distinct purposes in the near future.

Many useful sets can't be listed out entirely like the one above, but instead consist of elements with particular properties. We will write $\{x \mid x = 2n \text{ for } n \in \mathbb{Z}\}$ to mean the collection of all things that can be written as twice an integer – the collection usually called the even integers. We read this as "The set of all x's such that x equals two n for some n in the set of integers."

Definition: We say that a set A is a **subset** of a set B and write $A \subseteq B$ iff every element of A is also an element of B.

Example: If $A = \{1, 2, 3\}$, $B = \{0, 2\}$, and $C = \{2\}$, then we have $C \subseteq A$ and also $C \subseteq B$, but $B \not\subseteq A$ since $0 \in B$ but $0 \notin A$.

Definition: **Sets *A* and *B* are equal** iff $A \subseteq B$ and $B \subseteq A$.

Definition: The **empty set**, denoted $\varnothing$, is the set having no elements.

Definition: Given a set A, the **power set** of A, denoted $\mathcal{P}(A)$, is the set containing all subsets of A.

Example: If $A = \{a, b\}$, then $\mathcal{P}(A) = \{\varnothing, \{a\}, \{b\}, \{a, b\}\}$.

Exercises 2.1

1. For any set A, $\varnothing \subseteq A$.
2. For any set A, $A \subseteq A$.
3. If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.
4. For any set A, $A = A$.

5. Find $\mathcal{P}(\varnothing)$.

6. Find $\mathcal{P}(\{a\})$.

7. Find $\mathcal{P}(\{a, b, c\})$.

8. Find $\mathcal{P}(\{a, b, c, d\})$.

2.2 Operations on Sets

It's the way sets combine and build on each other that makes them useful.

Definition: If A and B are sets, then the **intersection** of A and B, denoted $A \cap B$, is

$$A \cap B = \{x \mid x \in A \text{ and } x \in B\}$$

Definition: If A and B are sets, then the **union** of A and B, denoted $A \cup B$, is

$$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$$

Definition: If A and B are sets, then the **set difference** of A and B, denoted $A \setminus B$ or $A - B$, is

$$A - B = \{x \mid x \in A \text{ and } x \notin B\}$$

Definition: If A is a set and X is the set of all elements under consideration, then the **complement** of A, denoted A', is

$$A' = \{x \mid x \in X \text{ and } x \notin A\}$$

Definition: If A and B are sets, we say they are **disjoint** iff $A \cap B = \varnothing$.

Exercises 2.2

1. $A \subseteq A \cup B$

2. $A \cap B \subseteq A$

3. $A \cup B = B \cup A$

4. $A \cap B = B \cap A$

5. $A \cup A' = X$

6. $A \cap A' = \varnothing$

7. $A - \varnothing = A$

8. $(A')' = A$

9. $(A \cap B)' = A' \cup B'$

10. $(A \cup B)' = A' \cap B'$

11. $A \cup (B \cup C) = (A \cup B) \cup C$

12. $A \cap (B \cap C) = (A \cap B) \cap C$

13. $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

14. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

15. If A and B are disjoint, and B and C are disjoint, then A and C are disjoint.

16. If A and B are disjoint, then $A - B = A$.

2.3 Arbitrary Unions and Intersections

If the union of two sets A_1 and A_2 is the collection of all elements that are in A_1 or A_2, it is natural to extend this to the union of three sets A_1, A_2, and A_3 being the collection of all elements that are in A_1 or A_2 or A_3. We can make this idea more general by letting I be a set (called the indexing set) of all of the subscripts involved, and saying that we're taking the union over all $i \in I$.

Definition: Let I be a set such that for each $i \in I$, A_i is itself a set. Then the **union of the A_i over I**, denoted $\bigcup_{i\in I} A_i$, is

$$\bigcup_{i\in I} A_i = \{x \mid x \in A_i \text{ for some } i \in I\}$$

Example: Suppose that $B_i = \{i, i + 1\}$ for each $i \in \mathbb{N}$, so for instance $B_7 = \{7, 8\}$. Then if we let E be the set of even natural numbers, $\bigcup_{i\in E} B_i$ would be the set of all natural numbers, since any even element $n \in \mathbb{N}$ would be in B_n itself, and any odd element n would be in B_{n-1}. On the other hand if we let T be the set of threven natural numbers, then $\bigcup_{i\in T} B_i$ would be the set of all naturals which are either threven or throdd (but not the throddodds).

Definition: Let I be a set such that for each $i \in I$, A_i is itself a set. Then the **intersection of the A_i over I**, denoted $\bigcap_{i\in I} A_i$, is

$$\bigcap_{i\in I} A_i = \{x \mid x \in A_i \text{ for all } i \in I\}$$

Exercises 2.3

1. For the sets B_i from the example above, what is $\bigcap_{i\in\mathbb{N}} B_i$?

2. For each $i \in \mathbb{N}$, let $C_i = \{n \mid n \in \mathbb{N} \text{ and } n \leq i\}$. What is $\bigcup_{i\in\{1,2,3\}} C_i$?

3. For each $i \in \mathbb{N}$, let $C_i = \{n \mid n \in \mathbb{N} \text{ and } n \leq i\}$. What is $\bigcup_{i\in\mathbb{N}} C_i$?

4. For each $i \in \mathbb{N}$, let $C_i = \{n \mid n \in \mathbb{N} \text{ and } n \leq i\}$. What is $\bigcap_{i\in\{1,2,3\}} C_i$?

5. For each $i \in \mathbb{N}$, let $C_i = \{n \mid n \in \mathbb{N} \text{ and } n \leq i\}$. What is $\bigcap_{i\in\mathbb{N}} C_i$?

6. Using $\mathbb{R}^+$ to denote the positive reals, let $D_x = \{y \mid -x \le y \le x\}$. What is $\bigcup_{x\in\mathbb{R}^+} D_x$?

7. Using $\mathbb{R}^+$ to denote the positive reals, let $D_x = \{y \mid -x \le y \le x\}$. What is $\bigcap_{x\in\mathbb{R}^+} D_x$?

8. For each $n \in \mathbb{N}$, let $E_n = \{x \in \mathbb{R} \mid 0 \le x \le n\}$. What is $\bigcup_{i\in\mathbb{N}} E_i$?

9. For each $n \in \mathbb{N}$, let $E_n = \{x \in \mathbb{R} \mid 0 \le x \le n\}$. What is $\bigcap_{i\in\mathbb{N}} E_i$?

10. If $j \in I$, then $A_j \subseteq \bigcup_{i\in I} A_i$

11. If $j \in I$, then $\bigcap_{i\in I} A_i \subseteq A_j$

12. $\left(\bigcap_{i\in I} A_i\right)' = \bigcup_{i\in I} A_i'$

13. $\left(\bigcup_{i\in I} A_i\right)' = \bigcap_{i\in I} A_i'$

14. $A \cup \bigcup_{i\in I} B_i = \bigcup_{i\in I} (A \cup B_i)$

15. $A \cap \bigcap_{i\in I} B_i = \bigcap_{i\in I} (A \cap B_i)$

16. $A \cup \bigcap_{i\in I} B_i = \bigcap_{i\in I} (A \cup B_i)$

17. $A \cap \bigcup_{i\in I} B_i = \bigcup_{i\in I} (A \cap B_i)$

2.4 Inequalities

Up to this point we have deliberately said as little as possible about what properties you must assume numbers have. Some essential properties of real numbers that we will take on faith for the moment concern how comparison interacts with addition and multiplication, how comparisons combine, and a property called trichotomy. None of these should be surprising – the point is that these are among the *only* things we intend to accept without further justification.

Comparison Addition Principle: If $a, b, c \in \mathbb{R}$, and $a < b$, then $a + c < b + c$.

Comparison Multiplication Principle: If $a, b, c \in \mathbb{R}$, with $c > 0$ and $a < b$, then $a \cdot c < b \cdot c$.

Transitive Property of Inequality: If $a, b, c \in \mathbb{R}$, with $a < b$ and $b < c$, then $a < c$.

Trichotomy: If a and b are real numbers, then either $a < b$, $a = b$, or $a > b$.

Also, as a matter of notation, we say $a < b$ interchangeably with $b > a$. Furthermore, we write $a \leq b$ iff $a < b$ or $a = b$, and similarly with $a \geq b$.

Exercises 2.4

1. Suppose that $a, b, c \in \mathbb{R}$. If $a < b$, then $a - c < b - c$.
2. Suppose that $a, b, c \in \mathbb{R}$. If $c < 0$ and $a < b$, then $a \cdot c > b \cdot c$.
3. Suppose that $a, b \in \mathbb{R}$. If $a < b$, then $a < \frac{a+b}{2} < b$.
4. Suppose that $a, b \in \mathbb{R}$. If $a < b$ then $a^2 < b^2$.
5. Suppose that $a, b \in \mathbb{R}$. If $a, b > 0$, then $a < b \Leftrightarrow a^2 < b^2$.
6. Suppose that $a, b \in \mathbb{R}$, with $a < b$ and $a, b > 0$. Then $\forall n \in \mathbb{N}$, $a^n \leq b^n$.
7. Suppose that $a, b \in \mathbb{R}$. If $a, b > 0$, then $a < b \Leftrightarrow \sqrt{a} < \sqrt{b}$.
8. Suppose that $a, b \in \mathbb{R}$. If $a, b > 0$, then $\sqrt{ab} \leq \frac{a+b}{2}$.
9. Suppose that $a, b \in \mathbb{R}$. If $a, b > 0$, then $\sqrt{a^2 + b^2} \leq a + b$.
10. Suppose that $a, b, c, d \in \mathbb{R}$, with $a < b$ and $c < d$. Then $a + c < b + d$.
11. Suppose that $a, b, c, d \in \mathbb{R}$, with $a < b$ and $c < d$. Then $a - c < b - d$.

12. Suppose that $a, b, c, d \in \mathbb{R}$, with $a < b$ and $c < d$. Then $ac < bd$.

13. Suppose that $a, b, c, d \in \mathbb{R}$, with $a < b$ and $c < d$ and $b, c > 0$. Then $ac < bd$.

14. Suppose that $a, b, c, d \in \mathbb{R}$, with $a < b$ and $c < d$. Then $a/c < b/d$.

15. Suppose that $a, b \in \mathbb{R}$. If $a^2 = b^2$, then $a = b$.

16. Suppose that r is a real number. Then $r^2 \geq r$ and $\dfrac{1}{r^2} \leq \dfrac{1}{r}$.

17. Suppose that r is a real number and $r \geq 1$. Then $r^2 \geq r$ and $\dfrac{1}{r^2} \leq \dfrac{1}{r}$.

2.5 Real Intervals

You've probably already had some experience in Calculus with intervals, but we treat them formally here and glimpse some hints of things to come.

Definition: For $a, b \in \mathbb{R}$, with $a < b$, the **open interval from *a* to *b***, denoted (a, b), is
$$(a, b) = \{x \mid x \in \mathbb{R} \text{ and } a < x < b\}$$

Definition: For $a \in \mathbb{R}$, the **open interval from *a* to ∞**, denoted (a, ∞), is
$$(a, \infty) = \{x \mid x \in \mathbb{R} \text{ and } a < x\}$$

Definition: For $a, b \in \mathbb{R}$, with $a < b$, the **closed interval from *a* to *b***, denoted $[a, b]$, is
$$[a, b] = \{x \mid x \in \mathbb{R} \text{ and } a \le x \le b\}$$

Definition: For $a \in \mathbb{R}$, the **closed interval from *a* to ∞**, denoted $[a, \infty)$, is
$$[a, \infty) = \{x \mid x \in \mathbb{R} \text{ and } a \le x\}$$

Definition: For $a, b \in \mathbb{R}$, with $a < b$, the **half-open intervals from *a* to *b***, denoted $(a, b]$ and $[a, b)$, are
$$(a, b] = \{x \mid x \in \mathbb{R} \text{ and } a < x \le b\}$$
$$[a, b) = \{x \mid x \in \mathbb{R} \text{ and } a \le x < b\}$$

Exercises 2.5

1. Let $A_n = (n, \infty)$ for each $n \in \mathbb{N}$. What is $\bigcup_{n\in\mathbb{N}} A_n$?
2. Let $A_n = (n, \infty)$ for each $n \in \mathbb{N}$. What is $\bigcap_{n\in\mathbb{N}} A_n$?
3. Let $\mathbb{N}^+ = \mathbb{N} - \{0\}$. Let $B_n = (-1/n, 1/n)$ for each $n \in \mathbb{N}^+$. What is $\bigcup_{n\in\mathbb{N}^+} B_n$?
4. Let $\mathbb{N}^+ = \mathbb{N} - \{0\}$. Let $B_n = (-1/n, 1/n)$ for each $n \in \mathbb{N}^+$. What is $\bigcap_{n\in\mathbb{N}^+} B_n$?
5. The intersection of two open intervals is an open interval.
6. The intersection of two open intervals is either an open interval or empty.
7. The intersection of finitely many open intervals is either an open interval or empty.
8. The intersection of arbitrarily many open intervals is either an open interval or empty.

2.6 Absolute Values

You're probably familiar with the absolute value function in a simple sense, where it basically means "Take off the negative sign if there is one." Our goal here is to give you some idea of how much more it is than that.

Definition: For any $x \in \mathbb{R}$, let $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$ be called the **absolute value function**.

Lemma 1: $\forall\, x \in \mathbb{R}, -|x| \leq x \leq |x|$.

Proof: Exercise 1.

Lemma 2: $\forall\, x, y \in \mathbb{R}$, $|x| \leq y$ iff $-y \leq x \leq y$.

Proof: Exercise 2.

Theorem (The Triangle Inequality): $\forall\, x, y \in \mathbb{R}, |x + y| \leq |x| + |y|$.

Proof: Well, by Proposition 1 we have that $-|x| \leq x \leq |x|$ and $-|y| \leq y \leq |y|$, and adding these gives us $-|x| - |y| \leq x + y \leq |x| + |y|$. If we rewrite this as $-(|x| + |y|) \leq x + y \leq |x| + |y|$ we can apply Proposition 2 to have $|x + y| \leq |x| + |y|$. □

One important use to which absolute values can be put is to express distance. Notice that if you consider two values like –3 and 5 on the number line, $|(-3) - (5)| = |-8| = 8$ gives the distance between those points. This leads to the following definition.

Definition: For $x, y \in \mathbb{R}$, define the distance from x to y, denoted $d(x, y)$, by

$$d(x, y) = |x - y|.$$

It can be shown that this definition leads to several properties we would hope distances would possess (see exercises).

Exercises 2.6

1. Prove Lemma 1.
2. Prove Lemma 2.
3. $\forall\, x \in \mathbb{R}, |x| \geq 0$.
4. $\forall\, x, y \in \mathbb{R}, |x \cdot y| = |x| \cdot |y|$
5. $\forall\, x, y \in \mathbb{R}, |x - y| \leq |x| - |y|$
6. $\forall\, x, y \in \mathbb{R}, |x| - |y| \leq |x - y|$

7. $\forall\, x \in \mathbb{R},\ d(x, x) = 0.$

8. $\forall\, x, y \in \mathbb{R},\ d(x, y) = 0 \Rightarrow x = y.$

9. $\forall\, x, y \in \mathbb{R},\ d(x, y) \geq 0.$

10. $\forall\, x, y \in \mathbb{R},\ d(x, y) = d(y, x).$

11. $\forall\, x, y, z \in \mathbb{R},\ d(x, y) \leq d(x, z) + d(z, y).$

2.7 Cross Products

You're familiar, at least in practical terms, with the idea of ordered pairs and treating ordered pairs of real numbers as points in a plane for purposes such as graphing functions. This section digs a bit more deeply into these ideas.

Definition: For sets A and B, define the **Cartesian product of *A* and *B***, denoted $A \times B$, by

$$A \times B = \{(a, b) \mid a \in A, b \in B\}.$$

Definition: Given two elements (a_1, b_1) and (a_2, b_2) in $A \times B$, we say $\boldsymbol{(a_1, b_1) = (a_2, b_2)}$ iff $a_1 = a_2$ and $b_1 = b_2$.

For instance, $\mathbb{R} \times \mathbb{R}$ is the familiar collection of all ordered pairs of real numbers, written in parentheses with a comma in between. We only consider two of these ordered pairs to be equal if both the first and second numbers match, which corresponds to our standard notion of two points being identical if they share both x and y coordinates. But this definition extends far beyond that familiar setting, covering possibilities both trivial and profound.

Example 1: If *Meat* = {beef, chicken, fish} and *Dessert* = {cake, pie, nothing}, then *Meat* × *Dessert* = {(beef, cake), (beef, pie), (beef, nothing), (chicken, cake), (chicken, pie), (chicken, nothing), (fish, cake), (fish, pie), (fish, nothing)}, and could be taken to be a list of possibilities when ordering main course and dessert at some (boring) restaurant. Notice that, as with any set, the order we listed the items isn't important – but it's convenient to list them in this order, since it makes it easy to check that everything has been included. None of this should seem too profound yet, but do recognize that contexts with little superficial connection (planes and dinner selections) have a common underlying mathematical structure.

Example 2: If we consider $\mathbb{R} \times \mathbb{R}$, then $A = (1,3]$ and $B = [1,2)$ are both subsets of $\mathbb{R}$, so it can be useful to depict the cross product $A \times B$ with the shaded region in the figure shown below. The dotted boundaries of the shaded region are conventionally used to represent open edges, so that points actually lying on those edges are not included but everything within is.

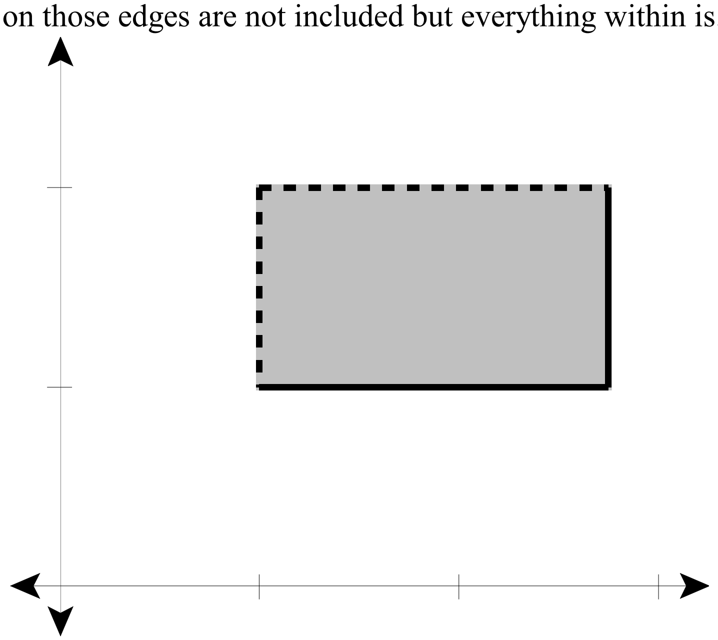

Figure 1

Exercises 2.7

1. For sets A and B, $A \times B = B \times A$.

2. For sets A, B, and C, $(A \times B) \times C = A \times (B \times C)$.

3. For sets A, B, C, and D, $A \subseteq B \wedge C \subseteq D \Rightarrow A \times C \subseteq B \times D$.

4. For sets A, B, C, and D, $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$.

5. For sets A, B, C, and D, $(A \cup B) \times (C \cup D) = (A \times C) \cup (B \times D)$.

2.8 Russell's Paradox

During the 20th century mathematicians and philosophers came to recognize some deep issues that arise from the casual use of sets. Many of these issues were dealt with to a certain degree, but in most cases only by giving up other things. Read *Gödel, Escher, Bach* or *Uncertainty* for more of the story.

Exercises 2.8

For these exercises, consider the set R consisting of all sets that are not elements of themselves.

1. Is $R \in R$?

2. Is $R \notin R$?

Appendix to Chapter 2

Proposition 1: Let A be some set and X be the universal set of all objects under consideration. Then $A \cup A' = X$.

Proof: Well, if we take $x \in A \cup A'$, we immediately know $x \in X$ since all objects under consideration are from X. Thus $A \cup A' \subseteq X$. Now on the other hand take $x \in X$. There are two cases to consider: either $x \in A$ or $\neg (x \in A)$. In the first case, since $x \in A$, we know $x \in A \cup A'$. In the second case, we have $x \in X$ and $\neg (x \in A)$, or $x \in X \wedge x \notin A$, so $x \in A'$, and therefore $x \in A \cup A'$. So in either case $x \in X$ implies $x \in A \cup A'$, so $X \subseteq A \cup A'$. Hence $A \cup A' = X$, as desired. □

Proposition 2: For sets A, B, and C, $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof: Well, take $x \in A \cap (B \cup C)$, so by definition $x \in A \wedge x \in B \cup C$. Then also by definition $x \in A \wedge (x \in B \vee x \in C)$. But then by the distributive property $(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$, so by definition $x \in (A \cap B) \cup (A \cap C)$. Thus $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$. Now if we take $x \in (A \cap B) \cup (A \cap C)$, by definition this means $(x \in A \wedge x \in B) \vee (x \in A \wedge x \in C)$. So there are two cases, so first suppose $x \in A \wedge x \in B$. We have $x \in A$, and since $x \in B$, we know $x \in B \cup C$, so $x \in A \cap (B \cup C)$. In the other case we suppose $x \in A \wedge x \in C$, so similarly we have $x \in A$, and since $x \in C$, we know $x \in B \cup C$, so $x \in A \cap (B \cup C)$. So in both cases we have $x \in A \cap (B \cup C)$, and thus $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Then we have shown $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. □