Examlet 1 Foundations of Advanced Math 2/3/12

1. Show that the product of two throddodd integers is throdd.

m is throddodd, so $m = 3p + 2$ $p \in \mathbb{Z}$

n is also throddodd, so $n = 3q + 2$ $q \in \mathbb{Z}$

$$m(n) = (3p+2)(3q+2) = 9pq + 6p + 6q + 4$$
$$= 9pq + 6p + 6q + 3 + 1$$
$$= 3(3pq + 2p + 2q + 1) + 1$$

Because $(3pq + 2p + 2q + 1)$ is also an integer by the closure of integers under multiplication and addition, the product of two throddodd integers is throdd because it matches the form of $3r+1$ that makes a number throdd.

Well done!

2. If a, b, and c are integers for which $a \mid b$ and $a \mid (b + c)$, then $a \mid c$.

$a|b$, a divides b can be rewritten as

$b = ax$ where $x \in \mathbb{Z}$

$a|(b+c)$, a divides $(b+c)$ can be rewritten as

$b+c = ay$ where $y \in \mathbb{Z}$

use substitution,

$ax + c = ay$

subtract ax from both sides

$c = ay - ax$

$c = a(y-x)$

Nice!

$(y-x)$ is an integer under the closure of $\mathbb{Z}$ under subtraction. a is a factor of c therefore a divides c, $a|c$. □

3. Determine whether $(P \wedge Q) \vee R$ is logically equivalent to $(P \vee R) \wedge (Q \vee R)$

Use a truth table:

P	Q	R	$(P \wedge Q)$	$(P \wedge Q) \vee R$	$(P \vee R)$	$(Q \vee R)$	$(P \vee R) \wedge (Q \vee R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	F	T	F	F
F	T	T	F	T	T	T	T
F	T	F	F	F	F	T	F
F	F	T	F	T	T	T	T
F	F	F	F	F	F	F	F

$(P \wedge Q) \vee R$ and $(P \vee R) \wedge (Q \vee R)$ have the same values under all circumstances therefore they are logically equivalent $\square$

Nice Job!

4. Show that if $a \equiv_n -1$, then $a^2 \equiv_n 1$.

$a \equiv_n -1$ can be rewritten as $n \mid -1-a$ which can also be rewritten as:

$pn = (-1-a)$ where $p \in \mathbb{Z}$.

rewrite: $pn + a = -1$

rewrite: $a = -1 - pn$

a^2 then equals $(-1-pn)^2$

simplify: $1 + pn + pn + p^2n^2$

Simplify: $1 + 2pn + p^2n^2$

Simplify: $1 + n(2p + p^2n)$

So $a^2 = 1 + n(2p + p^2n)$.

rewrite as: $n(-2p - p^2n) = 1 - a^2$

rewrite as: $n \mid 1 - a^2$

rewrite as: $a^2 \equiv_n 1$

Since $(2p + p^2n)$ is an element of the integers under the closure of multiplication and addition, $n(-2p - p^2n) = 1 - a^2$ can be rewritten as $n \mid 1 - a^2$ which too can be rewritten as $a^2 \equiv_n 1$.

Excellent!

5. For all $n \in \mathbb{N}$, $2^n \geq 1$.

For $n=0$, $2^0=1$ and $1 \geq 1$, so this statement is true in this case

Now, suppose that this statement is also true for $n=k$. This means that $2^k \geq 1$.

Because $2^n \geq 1$ $n \in \mathbb{N}$ is true for $n=k$, it should also be true for $n=k+1$.

If we start with our inductive hypothesis, we have $2^k \geq 1$. We can also say that $2^k \geq 0$ because $0<1$.

If we add the two inequalities together, we get $(2^k+2^k) \geq 1+0$.

This can be simplified to $(2^k)2 \geq 1$. It can be simplified further to $2^{(k+1)} \geq 1$. This shows that $2^n \geq 1$ when $n=k+1$. So, by Mathematical induction, $2^n \geq 1$ for all $n \in \mathbb{N}$.

Well done!