Examlet 1A   Foundations of Advanced Math   2/7/13

1. Show that the square of a throdd integer is throdd.

Proof: Let $n = 3m+1$ represent a throdd integer when m is an integer.
$n^2 = (3m+1)^2 = 9m^2 + 6m + 1 = 3(3m^2 + 2m) + 1$
By closure of integers, $3m^2 + 2m$ is an integer g.
$3g + 1$ is previously defined as throdd. $\square$

Good!

2. Determine whether $(P \Rightarrow Q) \wedge (P \Rightarrow R)$ is logically equivalent to $P \Rightarrow (Q \wedge R)$.

P	Q	R	P→Q	P→R	(P→Q) ∧ (P→R)	Q∧R	P→(Q∧R)
T	T	T	T	T	T	T	T
T	T	F	T	F	F	F	F
T	F	T	F	T	F	F	F
T	F	F	F	F	F	F	F
F	T	T	T	T	T	T	T
F	T	F	T	T	T	F	T
F	F	T	T	T	T	F	T
F	F	F	T	T	T	F	T

$(P \rightarrow Q) \wedge (P \rightarrow R)$ and $P \rightarrow (Q \wedge R)$ have the same truth values so they are logically equivalent. $\square$

Good!

11/

3. a) If $a \equiv_n b$, then $a + 2 \equiv_n b + 2$.

If $a \equiv_n b$, then $b - a = nk$ for k is some integer.

$b - a = nk$

$b - a + 2 = nk + 2$

$b + 2 - a - 2 = nk$

$b + 2 - (a + 2) = nk$

So since k is some integer $n \mid b + 2 - (a + 2)$

$\therefore\ a + 2 \equiv_n b + 2$. $\square$

Good.

b) If $a \equiv_n b$, then $2a \equiv_n 2b$.

If $a \equiv_n b$, then $b - a = nk$ for k is some integer.

$b - a = nk$

$2(b - a) = 2(nk)$

$2b - 2a = 2nk$

$2b - 2a = n(2k)$ by closure of integers $2k$ is some integer

so $n \mid 2b - 2a$

$\therefore\ 2a \equiv_n 2b$. $\square$

Nice!

4. $\sqrt{2}$ is irrational.

Proof: Suppose $\sqrt{2}$ is rational, so that $\sqrt{2} = \frac{p}{q}$ where p and q are both integers. (p and q have no common factors)
By squaring both sides, we get $2 = \frac{p^2}{q^2}$
$p^2 = 2q^2$
p^2 is even, so we know p is also even.
$p = 2n$ where n is some integer.

By substitution,

$(2n)^2 = 4n^2 = 2q^2$
$2n^2 = q^2$
q^2 is also even, so we know q is even as well.
If both p and q are even, they have a common factor of 2, going against our supposition that $\sqrt{2} = \frac{p}{q}$ where p and q have no common factors. $\sqrt{2}$ can't be written as a rational number and therefore by contradiction

Nice!

$\sqrt{2}$ is irrational. $\square$

5. For all $n \in \mathbb{N}$, $3 \mid (n^3 - n)$.

Proof: First, I am going to test for the base case $n=0$.

$0^3 - 0 = 0$

$3(0) = 0$. Since 0 is an integer, 3 divides 0 and the base case is true

Now, suppose this is also true for $n = k$

3 divides $k^3 - k$. So $3m = k^3 - k$ where m is some integer

Now, test for $k+1$.

3 divides $(k+1)^3 - (k+1)$

$(k^2 + 2k + 1)(k+1) - (k+1)$

$k^3 + 2k^2 + \cancel{k} + k^2 + 2k + \cancel{1} - \cancel{k} - \cancel{1}$

$k^3 + 3k^2 + 2k = k^3 + 3k^2 + 3k - k$

Substitute $3m = k^3 - k$

So now $3m + 3k^2 + 3k = 3(m + k^2 + k)$

Since $m + k^2 + k$ is an integer by closure of integers, 3 divides $(k+1)^3 - (k+1)$.

Since $3 \mid n^3 - n$ is true for the base case, and since supposing it's true for $n=k$ guarantees it's true for $n = k+1$, by mathematical induction $3 \mid n^3 - n$ for all $n \in \mathbb{N}$. $\square$

Nice!