Examlet 2a Foundations of Advanced Math 3/1/13

1. For any sets A, B, and C, $A \cup (B \cap C) \subseteq A \cup B$.

Let $x \in A \cup (B \cap C)$.

This means either $x \in A$ or $x \in B \cap C$, and if $x \in B \cap C$ then $x \in B$ and $x \in C$ by definition of intersects.

For the first case, $x \in A$, x would also be an element in $A \cup B$ since it is an element in A. For the second case, $x \in B \cap C$, x would also be an element in $A \cup B$ since $x \in B$.

$\therefore A \cup (B \cap C) \subseteq A \cup B$.

Good.

2. Suppose that $a, b, c \in \mathbb{R}$. If $c < 0$ and $a < b$, then $a \cdot c > b \cdot c$.

Proof: Take $c<0$. We can use the Comparison Addition Principle to add $-c$ to both sides, giving us $0<-c$. Now, we can take $a<b$ and use the Comparison Multiplication Principle to get $-ac<-bc$. Add the quantity $(ac+bc)$ to both sides (CAP) and you have $bc<ac$, which is the same as $ac>bc$. □

Excellent!

3. Let $\mathbb{R}^+ = \{x \mid x \in \mathbb{R} \text{ and } x > 0\}$. For each $x \in \mathbb{R}^+$, let $A_x = (0, x]$.

a) What is $\bigcap_{x\in\{1,2,3\}} A_x$?

$A_1 = (0, 1]$

$A_2 = (0, 2]$

$A_3 = (0, 3]$

$\bigcap_{x\in\{1,2,3\}} A_x = \underline{(0, 1]}$

b) What is $\bigcup_{x\in\{1,2,3\}} A_x$?

$A_1 = (0, 1]$

$A_2 = (0, 2]$

$A_3 = (0, 3]$

$\bigcup_{x\in\{1,2,3\}} A_x = \underline{(0, 3]}$

c) What is $\bigcap_{x\in\mathbb{R}^+} A_x$?

$\bigcap_{x\in\mathbb{R}^+} A_x = \underline{\emptyset}$

Good

d) What is $\bigcup_{x\in\mathbb{R}^+} A_x$?

$\bigcup_{x\in\mathbb{R}^+} A_x = \underline{(0, \infty)}$

4. Suppose I is a set and for each $i \in I$, A_i and B_i are sets, and that there is some $i \in I$ for which $A_i \subseteq B_i$.

a) Is it true that $\bigcup_{i \in I} A_i \subseteq \bigcup_{i \in I} B_i$? Support your answer.

Nope. S'pose $I = \{1, 2\}$, with $A_1 = \{1\}$, $A_2 = \mathbb{R}$,
$B_1 = \{1\}$, $B_2 = \{2\}$.

Then $\bigcup_{i \in I} A_i = \mathbb{R}$ and $\bigcup_{i \in I} B_i = \{1, 2\}$, so $\bigcup_{i \in I} A_i \not\subseteq \bigcup_{i \in I} B_i$.

But there is some $i \in I$, namely 1, for which $A_1 \subseteq B_1$.

b) Is it true that $\bigcap_{i \in I} A_i \subseteq \bigcap_{i \in I} B_i$? Support your answer.

Nope. S'pose $I = \{1, 2\}$, with $A_1 = \{1\}$, $A_2 = \{2\}$,
$B_1 = \{1\}$, $B_2 = \phi$.

So there is some $i \in I$, specifically 1, for which $A_1 \subseteq B_1$.

But $\bigcap_{i \in I} A_i = \{1, 2\}$ and $\bigcap_{i \in I} B_i = \phi$, so $\bigcap_{i \in I} A_i \not\subseteq \bigcap_{i \in I} B_i$.

5. $\forall x \in \mathbb{R}, -|x| \le x \le |x|$.

Case 1: $x \geq 0$, so by definition $|x| = x$, and we can also say $x \leq |x|$.
Adding $-x$ to both sides of $x \geq 0$ gives us $0 \geq -x$ or $-x \leq 0$.
Then we have $-|x| = -x \leq 0 \leq x \leq |x|$.

Case 2: $x < 0$, so by definition $|x| = -x$, and we can also say $-x \leq |x|$.
Adding $-x$ to both sides of $x < 0$ gives $0 < -x$.
Then we have $-|x| \leq x < 0 < -x \leq |x|$.

So in either case we have $-|x| \leq x \leq |x|$, as desired. $\square$