Examlet 4B Foundations of Advanced Math 4/19/13

1. a) State the definition of a reflexive relation.

A relation $\sim$ on a set S is reflexive iff $\forall a \in S$, $a \sim a$.

b) Give an example of a relation on the set $\{a, b, c\}$ which is transitive but not symmetric.

The relation $\{(a,b)\}$ is (vacuously) transitive, but not symmetric since it has (a,b) but not (b,a).

2. Let $S = \{1, 2, 3, 4, 5\}$, and consider the partition $\mathcal{P} = \{\{1,2\},\{3,5\},\{4\}\}$ of S. Write the equivalence relation $\sim$ corresponding to $\mathcal{P}$.

$\sim = \{(1,1),(1,2),(2,1),(2,2),(3,3),(3,5),(4,4),(5,3),(5,5)\}$

Great

3. a) Express the definition of the sum of two functions $f, g : \mathbb{R} \to \mathbb{R}$ formally in terms of ordered pairs.

$(x, y_1) \in f$ and $(x, y_2) \in g \Rightarrow (x, y_1 + y_2) \in f + g$

Great

b) Express the definition of a surjection formally in terms of ordered pairs.

$\forall b \in B, \exists a \in A$ such that $(a, b) \in f$

Nice!

4. Let S be a set and $\mathcal{P}$ a partition of S.

a) The relation on S defined by $a \sim b$ iff $\exists P \in \mathcal{P}$ for which $a, b \in P$ is a reflexive relation.

If $a \in S$, there must be at least one set in $\mathcal{P}$ that contains a because the sets of the partition of S form S when unioned together. Since $\exists P \in \mathcal{P}$ for which $a \in P$, $a \in P$. Since $a, a \in P$, $a \sim a$. Therefore, this is a reflexive relation.

Nice

b) The relation on S defined by $a \sim b$ iff $\exists P \in \mathcal{P}$ for which $a, b \in P$ is a symmetric relation.

Suppose $a, b \in S$ and $a \sim b$, meaning that $\exists P \in \mathcal{P}$ for which $a, b \in P$. Since $a, b \in P$, $b, a \in P$. This means that $b \sim a$. Therefore, the relation is symmetric.

Great

c) The relation on S defined by $a \sim b$ iff $\exists P \in \mathcal{P}$ for which $a, b \in P$ is a transitive relation.

Suppose $a, b, c \in S$, $a \sim b$, and $b \sim c$. This means that $\exists P \in \mathcal{P}$ such that $a, b \in P$, and $\exists Q \in \mathcal{P}$ such that $b, c \in Q$. By our definition of partition, $\mathcal{P}$ is pairwise disjoint, meaning that two sets in $\mathcal{P}$ can only share elements if they are equal. Therefore if $b \in P$ and $b \in Q$, $P = Q$. Since $P = Q$ and $a \in P$, $a \in Q$. $a, c \in Q$, so $a \sim c$. Therefore, this is a transitive relation.

Excellent!

5. Say that two vertices v_1 and v_2 of a graph G are **adjacent** iff there exists a walk with exactly one edge between them.

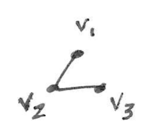
a) Is the relation of being adjacent reflexive?

Not necessarily. The graph: 
with no edges provides a counterexample, since there is no walk with any edges from v_1.

b) Is the relation of being adjacent symmetric?

Yes. If $v_1 \sim v_2$ that means there is a walk $v_1 e_1 v_2$ with exactly one edge. But reversing this gives us $v_2 e_1 v_1$, which is still a walk with exactly one edge, so $v_2 \sim v_1$ as well.

c) Is the relation of being adjacent transitive?

Not in general. In the graph  there is a walk with exactly one edge from v_1 to v_2, so $v_1 \sim v_2$, and a walk with exactly one edge from v_2 to v_3, so $v_2 \sim v_3$. But there is no walk with exactly one edge from v_1 to v_3, so $v_1 \not\sim v_3$, and thus the relation is not transitive.

Bonus: But note that in the graph pictured in part a above the relation is transitive (vacuously).