Problem Set 7.5 Foundations Due 4/3/13

You have a wonderful task ahead of you! All semester long, you have used definitions and propositions to prove things. Today you will also use your INTUITION and give a few conjectures without proof.

This is an optional problem set, meaning if you turn this in, you can replace your lowest problem set grade with the score for this. Four of these problems will be graded, with each problem worth 5 points. You are welcome to discuss these problems with anyone and everyone, but must write up your own final submission without reference to any sources other than the textbook and instructor.

Theorem A. For any real $r : |r| < 1$,

$$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

Theorem B. Any real number between 0 and 1 can be represented in a base-2 form,

$$(0.a_1a_2a_3a_4\ldots)_2$$

with $a_i \in \{0,1\}$, which can be rewritten as the (decimal) sum

$$a_1 \cdot \left(\frac{1}{2}\right)^1 + a_2 \cdot \left(\frac{1}{2}\right)^2 + \ldots + a_n \cdot \left(\frac{1}{2}\right)^n + \ldots$$

Example 1: Base-2 to decimal with repeating digits.

$$\begin{aligned}
(0.10\bar{1})_2 &= 1 \cdot \left(\frac{1}{2}\right)^1 + 0 \cdot \left(\frac{1}{2}\right)^2 + 1 \cdot \left(\frac{1}{2}\right)^3 + 1 \cdot \left(\frac{1}{2}\right)^4 + \ldots + 1 \cdot \left(\frac{1}{2}\right)^n + \ldots \\
&= \frac{1}{2} + \left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)^4 + \ldots + \left(\frac{1}{2}\right)^n + \ldots \\
&= \frac{1}{2} + \left(\frac{1}{2}\right)^3 \left(1 + \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \ldots + \left(\frac{1}{2}\right)^{n-3} + \ldots\right) \\
&= \frac{1}{2} + \left(\frac{1}{2}\right)^3 \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n \\
&= \frac{1}{2} + \left(\frac{1}{2}\right)^3 \frac{1}{1-1/2} \quad \text{by Theorem A} \\
&= \frac{1}{2} + \left(\frac{1}{2}\right)^3 \cdot 2 \\
&= \frac{1}{2} + \frac{1}{8} \cdot 2 \\
&= \frac{3}{4}
\end{aligned}$$

Example 2: Simple decimal to base-2, and non-uniqueness of base-2 representation.
Algorithm: multiply by 2, take the integer part, find the remainder, multiply it by 2, take the integer part, repeat; OR, subtract the greatest multiple of the smallest power $\left(\frac{1}{2}\right)$, then subtract multiples of the next smallest power $\left(\frac{1}{4}\right)$, repeat.

$$\begin{aligned}\frac{3}{4} &= 1 \cdot \frac{1}{2} + 1 \cdot \frac{1}{4} \\ &= (0.11)_2\end{aligned}$$

Theorem C. Any real number between 0 and 1 can be represented in a base-3 form,

$$(0.a_1a_2a_3a_4\ldots)_3$$

with $a_i \in \{0, 1, 2\}$, which can be rewritten as the (decimal) sum

$$a_1 \cdot \left(\frac{1}{3}\right)^1 + a_2 \cdot \left(\frac{1}{3}\right)^2 + \ldots + a_n \cdot \left(\frac{1}{3}\right)^n + \ldots$$

Example 3: Converting decimal to base-3.
Algorithm: multiply by 3, take the integer part, find the remainder, multiply it by 3, take the integer part, repeat; OR, subtract the greatest multiple of the smallest power $\left(\frac{1}{3}\right)$, then subtract multiples of the next smallest power $\left(\frac{1}{9}\right)$, repeat.

$$\begin{aligned}0.7 &= 2 \cdot 0.\overline{3} + 0.0\overline{3} \\ &= 2 \cdot \frac{1}{3} + 0 \cdot 0.\overline{1} + 0 \cdot 0.\overline{037} + 2 \cdot 0.\overline{012345679} + 0.\overline{0086419653} \\ &= 2 \cdot \frac{1}{3} + 0 \cdot \frac{1}{9} + 0 \cdot \frac{1}{27} + 2 \cdot \frac{1}{81} + 2 \cdot 0.\overline{0041152263} + 0.\overline{0004115127} \\ &= 2 \cdot \frac{1}{3} + 0 \cdot \frac{1}{9} + 0 \cdot \frac{1}{27} + 2 \cdot \frac{1}{81} + 2 \cdot \frac{1}{243} + \ldots \\ &= (0.2002200220022\ldots)_3 = (0.\overline{2002})_3\end{aligned}$$

For Problems 1-5, consider the *continuous* function $f(x)$ on the domain [0,1] with the properties:

$$f(0) = 0$$

$$f'(x) = \begin{cases} \frac{3}{2} & \text{if} \quad 0 < x < \frac{1}{2} \\ \frac{3}{4} & \text{if} \quad \frac{1}{2} < x < \frac{3}{4} \\ \frac{3}{8} & \text{if} \quad \frac{3}{4} < x < \frac{7}{8} \\ \vdots & \quad \vdots \\ \frac{3}{2^n} & \text{if} \quad 1 - \left(\frac{1}{2}\right)^{n-1} < x < 1 - \left(\frac{1}{2}\right)^{n} \end{cases}$$

1. **Circle one: $f(x)$ is (increasing/strictly increasing/decreasing/strictly decreasing).**

2. **Circle one: $f(x)$ is (bounded/unbounded).**

3. **Circle one: $f(x)$ is (injective/surjective/bijective/none of the above).**

4. **Define (piecewise) an odd function $g(x)$ which equals $f(x)$ on the domain of $f(x)$.**

5. **Rewrite the function definition of $f(x)$ using base-2 and the fact that $\frac{3}{2} = (1.1)_2$.**

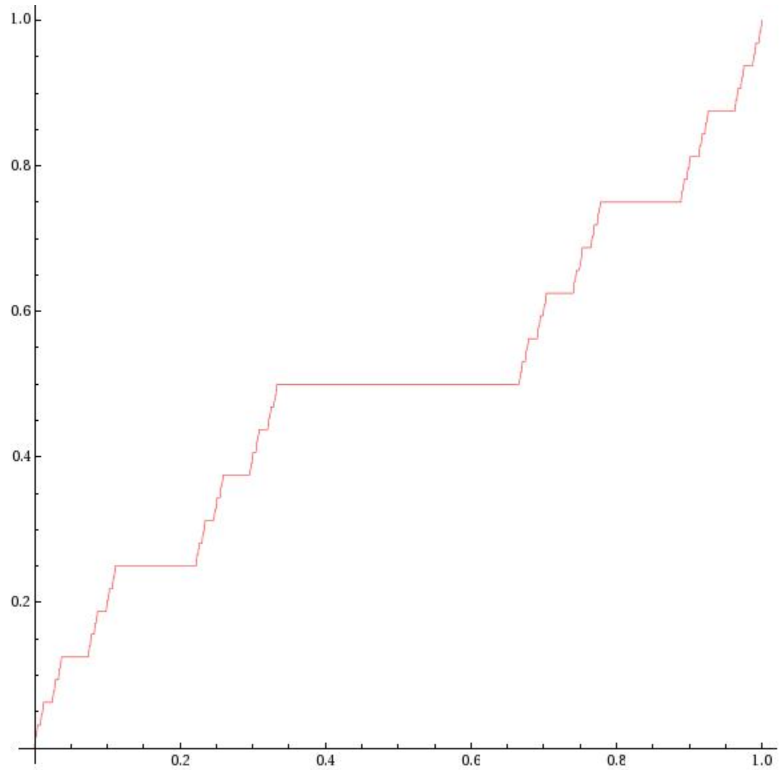
For Problems 6-9, consider the function $m(x)$ on the domain [0,1] with the construction:

i. Express x in base-3.

ii. If x contains a 1, replace every digit after the first 1 by 0.

iii. Replace all 2s with 1s.

iv. Interpret the result as a base-2 number. The result is $m(x)$.



What this function does, is maps the middle third of the domain (the 1 in base-3), constantly to the middle half of the codomain. It takes the middle third of the remaining thirds (the 0 and 2 in base-3, whose thirds are in the ninths-place) and maps them constantly to another part of the codomain. The process repeats, mapping smaller domains to smaller codomains. It is self-similar.

6. **Circle one:** $m(x)$ **is (increasing/strictly increasing/decreasing/strictly decreasing).**

7. **Circle one:** $m(x)$ **is (bounded/unbounded).**

8. **Circle one:** $m(x)$ **is (injective/surjective/bijective/none of the above).**

9. **What, if anything, do you conjecture about the derivative of** $m(x)$**?**