Examlet 1 Advanced Geometry 2/11/13

1. a) State the definition of a metric.

A metric is a function $D: \mathbb{P} \times \mathbb{P} \to \mathbb{R}$ s.t. 1) $D(P,Q) = D(Q,P)\ \forall P,Q$

2) $D(P,Q) \geq 0\ \forall P,Q$ 3) $D(P,Q) = 0$ iff $P = Q$.

Good.

b) State the definition of parallel lines.

Lines ℓ and m are parallel if there does not exist any point P s.t. P lies on both ℓ and m.

Great

c) State the definition of an angle.

An angle is the union of two nonopposite rays with the same endpoint.

Excellent

2. Show that if A and B are distinct points, then there exists a unique point M such that M is the midpoint of $\overline{AB}$.

Let A and B be distinct points and let $\overline{AB}$ be the segment with endpoints A and B (hyp). We know there exists a coordinate function of $\overleftrightarrow{AB}$ s.t. $f(A)=0$ and $f(B)=d$. Since the coordinate function is one-to-one, we know that every real number maps to a point of $\overleftrightarrow{AB}$, so let C be the point s.t. $f(C)=d/2$. Now we have $AC=|f(A)-f(C)|=|0-d/2|=d/2$ and $CB=|f(C)-f(B)|=|d/2-d|=d/2$. So A, B, C are collinear points s.t. $AC+CB=AB$, thus C is between A and B and since $AC=CB$ we know C is the midpoint of $\overline{AB}$. Thus we have proven the existence of the midpoint.

Now we must prove the uniqueness. So, suppose C' is also the midpoint of $\overline{AB}$. That is A, B, C' are collinear with $AC'=C'B$ and $AC'+C'B=AB$. So, we have $C'B=AC'=|f(A)-f(C')|=|0-f(C')|$ and we know $AB=d$ so $C'B=AC'=d/2$ so $|0-f(C')|=|f(C')|=d/2$, so we have $f(C)=f(C')$. But since the coordinate function is one-to-one it must be that $C=C'$, thus the midpoint is unique.

Wonderful.

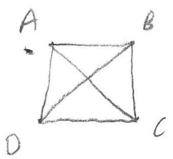
3. a) State the axioms for an incidence geometry.

A1: For every pair of distinct points A and B ∃ exactly one line ℓ s.t. both A and B lie on ℓ.

A2: For every line ℓ, ∃ at least two distinct points A and B s.t. both A and B lie on ℓ.

A3: ∃ three points that do not all lie on any one line

b) Suppose we have a set of points given by $\{A, B, C, D\}$ and our lines are the sets $\{A, B\}$, $\{A, C\}$, $\{A, D\}$, $\{B, C\}$, $\{B, D\}$, and $\{C, D\}$. Does this form an incidence geometry? Explain.



Yes. It satisfies all of the Axioms

A1: Let A and B be any pair of points, ∃ exactly one line s.t. both A and B lie on that line

A2: Every line has two points that lie on it

A3: there are three points that do not all lie on any one line

c) Suppose we have a set of points given by $\{A, B, C, D\}$ and our lines are the sets $\{A, B, C\}$, $\{A, B, D\}$, $\{A, C, D\}$, and $\{B, C, D\}$. Does this form an incidence geometry? Explain.

No. Axiom 1 says for every pair of distinct points

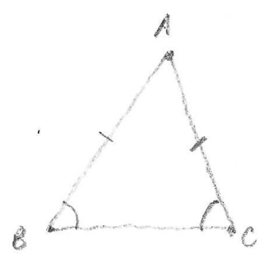
A and B ∃ exactly one line ℓ s.t. both A and B lie on ℓ.

But in this geometry points A and B lie on two distinct lines, (A and D, B and D, C and D, A and C, B and C also lie on two lines.

Excellent!

4. Prove that the base angles of an isosceles triangle are congruent.

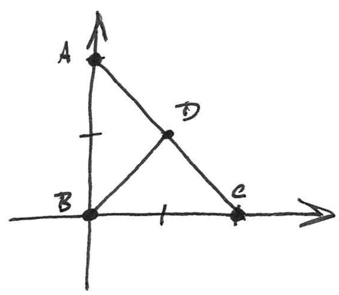
Let $\triangle ABC$ be a triangle with $\overline{AB} \cong \overline{AC}$. Then $\triangle BAC \cong \triangle CAB$ by S.A.S. So $\angle ABC \cong \angle ACB$ by definition of congruent triangles



Excellent!

5. Explain why taxicab geometry is important in regard to the SAS Postulate.

Taxicab geometry provides a model to demonstrate that SAS is independant of our previous 5 postulates. Both taxicab geometry and Cartesian geometry satisfy the first 5 postulates, but Cartesian geometry also satisfies SAS. Taxicab geometry, however, does not - consider the two triangles $\triangle ABC$ and $\triangle BDC$ shown below, where $A=(0,2)$, $B=(0,0)$, $C=(2,0)$, and $D=(1,1)$.



We have $AB=2$, $BC=2$, and the angle between them $90°$. We also have $BD=2$, $DC=2$, and the angle between them $90°$. Thus by SAS (if it applied), the triangles would be congruent.

However, it is easy to see that since $CA=4$ and $CB=2$, in fact these triangles are not congruent. Thus SAS is not a legitimate way of determining whether triangles are congruent in Taxicab geometry. Or to say it differently, we have two models, Cartesian and Taxicab, which satisfy our first five axioms. One also satisfies SAS, but the other doesn't, so SAS is consistent with and independant from the first five.