**Exam 2 Differential Equations 3/21/14**

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. Does the system of differential equations

$$\frac{dx}{dt} = 3x + y$$
$$\frac{dy}{dt} = -2x$$

have $x(t) = e^{2t}$, $y(t) = -2e^{2t}$ as a solution?

$x'(t) = 2e^{2t}$
$y'(t) = -4e^{2t}$

$2e^{2t} \overset{?}{=} 3e^{2t} - 2e^{2t}$ NOT EQUAL
$= e^{2t}$

$-4e^{2t} = -2e^{2t}$ NOT EQUAL

Since we plugged it in and it didn't work, this is not a solution.

Excellent!

2. State the definition of the Laplace transform for a function $y(t)$ with at most exponential growth.

The Laplace transform for a function $y(t)$ can be defined as

$$\mathcal{L}[y(t)] = \int_0^\infty y(t)\, e^{-st}\, dt$$

for all values of s for which the improper integral converges.

Great!

3. Construct a system of differential equations, with all coefficients representing positive constants, to model the interaction of two populations where:
   - The first population would experience logistic growth with carrying capacity $K$ in the absence of the second
   - Interaction between the two populations hurts the first population
   - The second population would experience exponential decline in the absence of the first
   - Interaction between the two populations benefits the second population
   - A fixed number of the second population are harvested in each unit of time

$$\frac{dx}{dt} = \alpha_1 x\left(1 - \frac{x}{K}\right) - \alpha_2 xy$$

$$\frac{dy}{dt} = -\beta_1 y + \beta_2 xy - \mu$$

where, $\alpha_1 \to$ constant of proportionality for logistic growth of $x$

Excellent!

$\alpha_2 \to$ constant of hurt to $x$ due to interaction

$\beta_1 \to$ constant of exponential decline of $y$

$\beta_2 \to$ constant of benefit to $y$ from interaction.

$\mu \to$ harvested $y$

4. Consider the system
$$\frac{dR}{dt} = 2\left(1 - \frac{R}{3}\right)R - RF$$
$$\frac{dF}{dt} = -16F + 4RF$$
. Find all equilibrium points of this system.

$0 = 2R - \frac{2}{3}R^2 - RF$ or $0 = R(2 - \frac{2}{3}R - F)$

$0 = 4F(-4 + R)$

So $F = 0$ or $R = 4$

$\Downarrow$

$R = 0$ or $2 - \frac{2}{3}R = 0$

$R = 3$

$\Rightarrow$

$2 - \frac{2}{3}(4) - F = 0$

$-\frac{2}{3} = F$

So $(0,0)$
$(3,0)$
$(4, -\frac{2}{3})$
are the equilibrium points

5. Consider the system $\begin{aligned}\frac{dx}{dt} &= x+2y \\ \frac{dy}{dt} &= -y\end{aligned}$. Use Euler's method with a step size of $\Delta t = 0.5$ to project $x(1)$ if $x(0) = 2$ and $y(0) = 3$.

| t | x | y |
|---|---|---|
| 0 | 2 | 3 |
| .5 | 6 | 1.5 |
| 1 | 10.5 | .75 |

$x'(0) = 2+2(3) = 8$ $\quad y'(0) = -3$

$x(.5) \approx 2 + 8(.5) \approx 2+4 = 6$ $\quad y(.5) \approx 3 - 3(.5) \approx 1.5$

$x'(.5) = 6+3 = 9$ $\quad y'(.5) = -1.5$

$x(1) \approx 6 + 9(.5) \approx 6+4.5 \approx 10.5$ $\quad y(1) \approx 1.5 - (1.5)(.5) \approx .75$

$\boxed{\begin{aligned} x(1) &\approx 10.5 \\ y(1) &\approx .75\end{aligned}}$

Great

6. What is the Laplace transform of $y(t) = 0$?

$$\mathcal{L}(y) = \int_0^\infty y e^{-st}\,dt$$

$$= \int_0^\infty 0 \cdot e^{-st}\,dt$$

$$= 0 \int_0^\infty e^{-st}\,dt$$

$$= 0$$

Nice

7. Let $y(t) = 5t$. Compute the Laplace transform of $y(t)$ from the definition.

$$\mathcal{L}[y(t)] = \int_0^\infty y(t)\, e^{-st}\, dt$$

$$= \int_0^\infty 5t\, e^{-st}\, dt$$

$$= 5 \int_0^\infty t\, e^{-st}\, dt$$

$$\begin{array}{c|l} + \; t & e^{-st} \\ - \; 1 & -\tfrac{1}{s} e^{-st} \\ + \; 0 & \tfrac{1}{(-s)^2} e^{-st} \end{array}$$

$$= \lim_{b \to \infty} 5 \left[ -\frac{t}{s} e^{-st} - \frac{1}{s^2} e^{-st} \right]_0^b$$

$$= 5 \left[ \left( \underbrace{-\frac{b}{s} e^{-sb}} - \underbrace{\frac{1}{s^2} e^{-sb}} \right) - \left( \underbrace{-\frac{0}{s} e^{-s(0)}}_{=0} - \frac{1}{s^2} \cancel{e^{-s(0)}}^{=1} \right) \right]$$

go to zero assuming $s > 0$

$$= 5\left(\frac{1}{s^2}\right)$$

$$\boxed{= \frac{5}{s^2}} \quad \text{for } \underline{s > 0}$$

*Wonderful!*

$\star$ Also know
$\mathcal{L}[t] = \frac{1}{s^2}$ from homework
so $\mathcal{L}[5t] = 5\mathcal{L}[t]$
$= \frac{5}{s^2}$

8. Consider the system $\begin{aligned} \frac{dR}{dt} &= -\tfrac{1}{2}F + 5R \\ \frac{dF}{dt} &= \quad\quad 8R \end{aligned}$. Find a solution to this system.

$$\frac{d^2R}{dt^2} = 5\frac{dR}{dt} - \frac{1}{2}(8R)$$

$$R'' = 5R' - 4R$$

$$R'' - 5R' + 4R = 0$$

Suppose $R = e^{st}$, then $R' = se^{st}$ and $R'' = s^2e^{st}$

$$s^2e^{st} - 5se^{st} + 4e^{st} = 0$$

$$e^{st}(s^2 - 5s + 4) = 0$$

Since $e^{st} \neq 0$

$$s^2 - 5s + 4 = 0$$

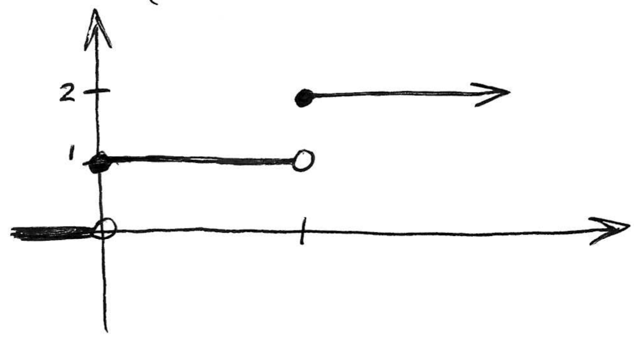
$$(s-4)(s-1) = 0$$

$s = 4$ or $s = 1$

$$\boxed{\begin{aligned} R &= k_1e^{4t} + k_2e^{t} \\ F &= 2k_1e^{4t} + 8k_2e^{t} \end{aligned}}$$

Lovely!

9. Let $y(t)=\begin{cases}0 & \text{if } t<0\\ 1 & \text{if } 0\le t<10\\ 2 & \text{if } 10\le t\end{cases}$. What is the Laplace transform of $y$?




That's like the sum of two Heaviside functions!

$$y(t) = u_0(t) + u_{10}(t)$$

So $\mathcal{L}(y(t)) = \mathcal{L}(u_0(t)) + \mathcal{L}(u_{10}(t))$

$$= \frac{e^{-0s}}{s} + \frac{e^{-10s}}{s}$$

$$= \frac{1}{s} + \frac{1}{s\cdot e^{10s}}$$

10. Suppose that $(x^*(t), y^*(t))$ is a solution to the system of differential equations
$$\frac{dx}{dt} = \alpha x + \beta y$$
$$\frac{dy}{dt} = \gamma x + \delta y$$
.

Is it possible to say whether $(3x(t), 3y(t))$ is a solution or not? Be clear about your reasoning.

$$\frac{dx}{dt} = \frac{dx^*(t)}{dt} = \alpha x + \beta y$$
$$\frac{dy}{dt} = \frac{dy^* t}{dt} = \gamma x + \delta y$$
since $x^*$ & $y^*$ are solutions

$$\frac{d(3x^*)}{dt} = 3\frac{dx^*}{dt}$$
$$\frac{d(3y^*)}{dt} = 3\frac{dy^*}{dt}$$
by properties of derivatives

$$\alpha 3x + \beta 3y = 3(\alpha x + \beta y) = 3\frac{dx}{dt} = 3\frac{dx^*}{dt}$$
$$\gamma 3x + \delta 3x = 3(\gamma x + \delta y) = 3\frac{dy}{dt} = 3\frac{dy^*}{dt}$$

since $\frac{dx}{dt} = \frac{dx^*}{dt}$ & $\frac{dy}{dt} = \frac{dy^*}{dt}$

yes $3x(t)$ & $3y(t)$ would be a solution

Exactly!