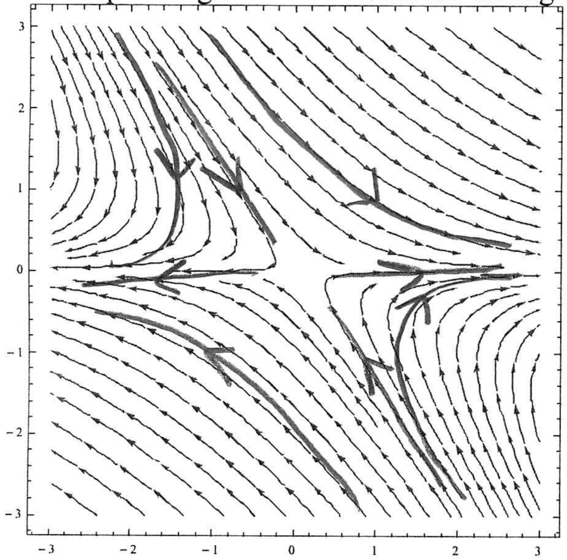
Exam 3 Differential Equations 4/18/14

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

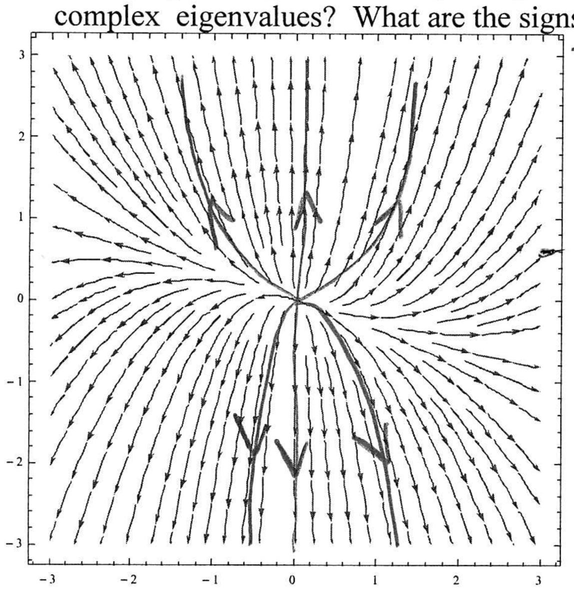
1. Does the planar system whose phase plane is shown below have two real eigenvalues, or two complex eigenvalues? What are the signs on the real portions of the eigenvalues?



This is a saddle which means $\lambda_1 < 0 < \lambda_2$ and both are real eigenvalues but one is positive and the other is negative.

Great

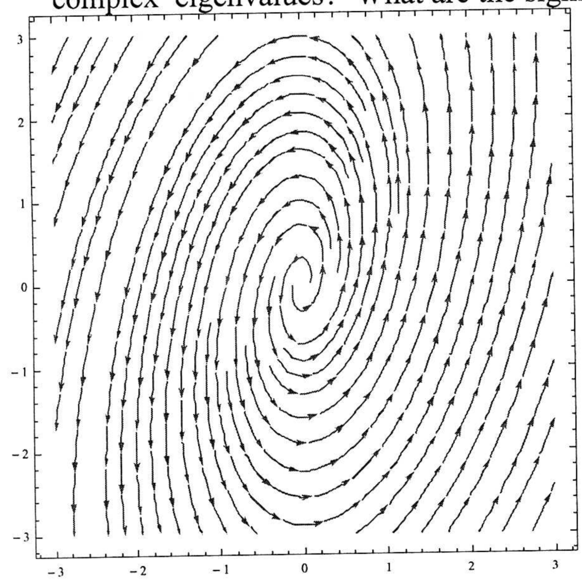
2. Does the planar system whose phase plane is shown below have two real eigenvalues, or two complex eigenvalues? What are the signs on the real portions of the eigenvalues?



This is a source which means $0 < \lambda_1 < \lambda_2$ and both are real eigenvalues and both are positive.

Nice

3. Does the planar system whose phase plane is shown below have two real eigenvalues, or two complex eigenvalues? What are the signs on the real portions of the eigenvalues?



The plane has two complex eigenvalues

the real portion is positive since the spiral appears to be a source.

Good

4. If a planar system of differential equations has eigenvalues $\lambda_1 = -4$, $\lambda_2 = 2$ and associated eigenvectors $\mathbf{v}_1 = (1,2)$ and $\mathbf{v}_2 = (-1,1)$, write a general solution to the system.

$$\hat{Y}(t) = Ae^{\lambda_1 t}\vec{v}_1 + Be^{\lambda_2 t}\vec{v}_2$$

$$\boxed{\hat{Y}(t) = Ae^{-4t}\begin{pmatrix}1\\2\end{pmatrix} + Be^{2t}\begin{pmatrix}-1\\1\end{pmatrix}}$$ a general solution

Good

5. Find the solution (in scalar form) of the initial-value problem $\frac{d^2y}{dt^2}+5\frac{dy}{dt}+6y=0$, with $y(0)=0$, $y'(0)=3$.

$y''+5y'+6y=0$ suppose $y=e^{st}$

$y'=se^{st}$

$y''=s^2e^{st}$

Substitute

$s^2e^{st}+5se^{st}+6e^{st}=0$

$e^{st}(s^2+5s+6)=0$

$e^{st}\neq 0$

$s^2+5s+6=0$

$(s+2)(s+3)$

$s=-2,-3$

Great

$y(t)=Ae^{-2t}+Be^{-3t}$ general solution

$y'(t)=-2Ae^{-2t}-3Be^{-3t}$

$y(0)=A+B=0$

$y'(0)=-2A-3B=3$

$-2(3)-3(-3)=3$

$-6+9=3$

so $\boxed{y(t)=3e^{-2t}-3e^{-3t}}$ particular solution

6. Consider the system $\frac{d\mathbf{Y}}{dt}=\begin{pmatrix}3 & 4\\ 1 & 0\end{pmatrix}\mathbf{Y}$. Find a general solution to this system.

$\det\begin{pmatrix}3-\lambda & 4\\ 1 & -\lambda\end{pmatrix}=(3-\lambda)(-\lambda)-4=0$

$\lambda^2-3\lambda-4$

$(\lambda-4)(\lambda+1)$

$\lambda=4 \quad \lambda=-1$

if $\lambda=4$

$3x+4y=4x \Rightarrow 4y=x \quad \begin{pmatrix}4\\1\end{pmatrix}=V_1$

if $\lambda=-1$

$3x+4y=-x \Rightarrow 4y=-4x \quad \begin{pmatrix}1\\-1\end{pmatrix}V_2$

So by the bandicoot theorem and Linearity Principle,

$Y(t)=Ae^{4t}\begin{pmatrix}4\\1\end{pmatrix}+Be^{-t}\begin{pmatrix}1\\-1\end{pmatrix}$ is the general solution.

Excellent!

7. State and prove the Bandicoot Theorem.

For the differential equation $\frac{d\vec{Y}}{dt} = \hat{A}\hat{Y}$, where λ is an eigenvalue of matrix $\hat{A}$ with corresponding eigenvector $\vec{v}$, $\hat{Y} = e^{\lambda t}\vec{v}$ is a solution.

Proof: If $\hat{Y} = e^{\lambda t}\vec{v}$

then
$$\frac{d\hat{Y}}{dt} = \lambda e^{\lambda t}\vec{v}$$
$$= \lambda \vec{v} e^{\lambda t}$$
$$\left.\begin{aligned} &= \hat{A}\vec{v}e^{\lambda t} \\ &= \hat{A}e^{\lambda t}\vec{v} \\ &= \hat{A}\hat{Y} \end{aligned}\right\} \text{By definition of eigenvalues and eigenvectors}$$

Since we plugged it in and it worked, $\hat{Y} = e^{\lambda t}\vec{v}$ is a solution to $\frac{d\hat{Y}}{dt} = \hat{A}\hat{Y}$.

Nice!

8. Consider the system $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\mathbf{Y}$. Find a solution to this system satisfying the initial condition $\mathbf{Y}(0) = (1,0)$.

$$\det\begin{pmatrix} 2-\lambda & 1 \\ -1 & 4-\lambda \end{pmatrix} = 0$$

$$(2-\lambda)(4-\lambda) - (-1)(1) = 0$$

$$\lambda^2 - 2\lambda - 4\lambda + 8 + 1 = 0$$

$$\lambda^2 - 6\lambda + 9 = 0$$

$$(\lambda-3)(\lambda-3) = 0$$

$\underline{\lambda = 3}$ ← repeated eigenvalue!

Excellent!

$$\boxed{\hat{Y}(t) = e^{3t}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + te^{3t}\begin{pmatrix} -1 \\ -1 \end{pmatrix}}$$

For repeated eigenvalues use The Great Theorem of page 305.

$$\hat{Y}(t) = e^{\lambda t}\vec{V}_0 + te^{\lambda t}\vec{V}_1$$

$$\vec{V}_1 = (\hat{A} - \lambda\hat{I})\vec{V}_0$$

$$\overset{\hat{A}}{\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}} - \overset{\lambda I}{\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\overset{\vec{V}_0}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} = \begin{pmatrix} -1+0 \\ -1+0 \end{pmatrix} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

$$\vec{V}_1 = \begin{pmatrix} -1 \\ -1 \end{pmatrix}$$

9. Consider the system $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -1 & \overset{b}{\cancel{-2}} \\ 1 & -4 \end{pmatrix}\mathbf{Y}$. If Jon wants to alter this system slightly in order *not* to have any distinguished lines in the phase plane, what values could Jon replace the –2 in the top right corner with to accomplish this?

$(-1-\lambda)(-4-\lambda) - b = 0$

$4 + \lambda + 4\lambda + \lambda^2 - b = 0$

$\lambda^2 + 5\lambda + 4 - b = 0$

In order not to have any distinguished lines in the phase plane, we need to have complex eigenvalues.

So, $\frac{-5 \pm \sqrt{25 - 4(4-b)}}{2}$. In order for our answer to be complex, $25 - 4(4-b) < 0$

$25 - 16 + 4b < 0$

$9 + 4b < 0$

$4b < -9$

$\boxed{b < -9/4}$

So Jon could replace -2 with any value less than -9/4.

Nice Job!

Check: $b = -3$

$\frac{-5 \pm \sqrt{25 - 4(4-(-3))}}{2} = \frac{-5 \pm \sqrt{-3}}{2} = \frac{-5 \pm i\sqrt{3}}{2}$ so complex; will not have distinguished lines + will be a spiral sink.

10. Find the general solution to the linear system $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 3 \\ -2 & 2 \end{pmatrix}\mathbf{Y}$, and then the particular solution satisfying $\mathbf{Y}(0) = \begin{pmatrix} 3 \\ 2 \end{pmatrix}$.

$$(-2-\lambda)(2-\lambda) - (3)(-2) = 0$$
$$\lambda^2 + 2\lambda - 2\lambda - 4 + 6 = 0$$
$$\lambda^2 + 2 = 0$$
$$\lambda = \pm i\sqrt{2}$$

If $\lambda = i\sqrt{2}$:

$$-2x + 3y = i\sqrt{2}x$$
$$3y = (i\sqrt{2} + 2)x$$
$$\vec{v} = \begin{pmatrix} 3 \\ 2+i\sqrt{2} \end{pmatrix}$$

So a solution is $\hat{Y} = e^{i\sqrt{2}t}\begin{pmatrix} 3 \\ 2+i\sqrt{2} \end{pmatrix} = \left(\cos\sqrt{2}t + i\sin\sqrt{2}t\right)\begin{pmatrix} 3 \\ 2+i\sqrt{2} \end{pmatrix}$

$$= \begin{pmatrix} 3\cos\sqrt{2}t + i\,3\sin\sqrt{2}t \\ 2\cos\sqrt{2}t + i\,2\sin\sqrt{2}t + i\sqrt{2}\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{pmatrix}$$

$$= \begin{pmatrix} 3\cos\sqrt{2}t \\ 2\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{pmatrix} + i\begin{pmatrix} 3\sin\sqrt{2}t \\ 2\sin\sqrt{2}t + \sqrt{2}\cos\sqrt{2}t \end{pmatrix}$$

And by the C.M.T. a general solution is

$$\hat{Y} = A\begin{pmatrix} 3\cos\sqrt{2}t \\ 2\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{pmatrix} + B\begin{pmatrix} 3\sin\sqrt{2}t \\ 2\sin\sqrt{2}t + \sqrt{2}\cos\sqrt{2}t \end{pmatrix}$$

Conveniently we notice $A = 1$, $B = 0$ satisfies the initial condition, so

$$\hat{Y} = \begin{pmatrix} 3\cos\sqrt{2}t \\ 2\cos\sqrt{2}t - \sqrt{2}\sin\sqrt{2}t \end{pmatrix}$$ is our particular solution.