Examlet 2 Foundations of Advanced Math 2/28/14

1. For any sets A, B, C and D, if $A \subseteq B$ and $C \subseteq D$, then $A \cup C \subseteq B \cup D$.

Let $x \in A \cup C$, so $x \in A$ or $x \in C$.
In the case $x \in A$, we know $A \subseteq B$, meaning every element of A is also an element of B, so $x \in B$ as well. If $x \in B$, $x \in B \cup D$ as well, so $A \cup C \subseteq B \cup D$.
In the case $x \in C$, we know $C \subseteq D$, meaning every element of C is also an element of D, so $x \in D$ as well. If $x \in D$, $x \in B \cup D$ as well. $A \cup C \subseteq B \cup D$.
So, in either case, $A \cup C \subseteq B \cup D$ as desired. $\square$

Great

2. a) Suppose that $a, b, c, d \in \mathbb{R}$. If $a > b$ and $c > d$, then $a + c > b + d$.

We know $a>b$.
By CAP, $a+c>b+c$.
We also know $c>d$.
By CAP, $c+b>d+b$.
Combining these, we see $a+c>b+c>d+b$
and by the transitive property, $a+c>b+d$. □

Side note
$b+c=c+b$

Great

b) Suppose that $a, b, c, d \in \mathbb{R}$. If $a > b$ and $c > d$, then $a - c > b - d$.

This is not generally true.
Let $a=2$, $b=-1$, $c=6$, and $d=-3$.
$a>b$ as $2>-1$, and $c>d$ as $6>-3$.
However, $a-c \not> b-d$ as

$2-6 \not> -1--3$

$-4 \not> 2$.

So we have reached a counter example.

Excellent

3. For each $x \in \mathbb{N}$, let $A_n = (-1, n]$.

a) What is $\bigcap_{n\in\{1,2,3\}} A_n$?

$A_1 = (-1, 1]$

$A_2 = (-1, 2]$

$A_3 = (-1, 3]$

$$\bigcap_{n\in\{1,2,3\}} A_n = (-1, 1]$$

b) What is $\bigcup_{n\in\{1,2,3\}} A_n$?

$$\bigcup_{n\in\{1,2,3\}} A_n = (-1, 3]$$

c) What is $\bigcap_{n\in\mathbb{N}} A_n$?

$A_0 = (-1, 0]$ is contained in all the others, but nothing outside A_0 is in all of them, so

$$\bigcap_{n\in\mathbb{N}} A_n = (-1, 0]$$

d) What is $\bigcup_{n\in\mathbb{N}} A_n$?

Everything to the right of -1 is in here, since for any such real number, there's an $n \in \mathbb{N}$ bigger, so A_n contains that real number, so

$$\bigcup_{n\in\mathbb{N}} A_n = (-1, \infty)$$

4. $\forall x \in \mathbb{R}, -|x| \leq x \leq |x|$.

Case 1: $x<0$

By definition of absolute value, $|x|=-x$ which can be written as $|x| \geq -x$. Also $-|x|=x$ can be written as $-|x| \leq x$. Then $x<0$, so add $-x$ to both sides (by CAP) to get $0<-x$. Then, combining all of these, we have $-|x| \leq x < 0 < -x \leq |x|$, and by the transitive prop. of inequality, $-|x| \leq x \leq |x|$.

Case 2: $x \geq 0$

By def. of absolute value, $|x|=x$ which can be written as $|x| \geq x$. Also, $-|x|=-x$ can be written as $-|x| \leq -x$. Then $x \geq 0$, so adding $-x$ to both sides gives $0 \geq -x$ (by CAP). Combining all of these, $-|x| \leq -x \leq 0 \leq x \leq |x|$, and using the transitive property of inequality, $-|x| \leq x \leq |x|$.

Therefore, since this is true for all possible cases, we can conclude that $-|x| \leq x \leq |x|$. $\square$

Well done.

5. Let I be a set and for each $i \in I$ let A_i be a set. Then $\left(\bigcup_{i\in I} A_i\right)' = \bigcap_{i\in I} A_i'$.

Let $x \in \left(\bigcup_{i\in I} A_i\right)'$

$\Rightarrow \quad x \in X \wedge \neg(\exists i \in I : x \in A_i)$

$\Rightarrow \quad x \in X \wedge (\forall i \in I, \neg(x \in A_i))$

$\Rightarrow \quad x \in X \wedge (\forall i \in I ; x \notin A_i)$

$\Rightarrow \quad (\forall i \in I ; (x \in X) \wedge (x \notin A_i))$

$\Rightarrow \quad \forall i \in I ; x \in A_i'$

$\Rightarrow \quad x \in \bigcap_{i\in I} A_i'$

Therefore $\left(\bigcup_{i\in I} A_i\right)' \subseteq \bigcap_{i\in I} A_i'$ (1)

Let $x \in \bigcap_{i\in I} A_i'$

$\Rightarrow \quad \forall i \in I, x \in A_i'$

$\Rightarrow \quad \forall i \in I, ((x \in X) \wedge \neg(x \in A_i))$

$\Rightarrow \quad x \in X \wedge (\forall i \in I, \neg(x \in A_i))$

$\Rightarrow \quad x \in X \wedge \neg(\exists i \in I, x \in A_i)$

$\Rightarrow \quad x \in \left(\bigcup_{i\in I} A_i\right)'$

Therefore $\bigcap_{i\in I} A_i' \subseteq \left(\bigcup_{i\in I} A_i\right)'$ (2)

From (1) and (2), $\left(\bigcup_{i\in I} A_i\right)' = \bigcap_{i\in I} A_i'$

Good Job