Examlet 3 Foundations of Advanced Math 3/28/14

1. a) State the definition of an injection.

$f: A \to B$ is an injection iff $f(a) = f(b) \Rightarrow a = b$ for $a, b \in A$.

b) State the definition of a surjection.

$f: A \to B$ is a surjection iff $\forall b \in B, \exists a \in A$ such that $f(a) = b$.

c) State the definition of equipollent sets.

set A is equipollent to set B iff $\exists$ a bijection from A to B.

d) State the definition of a denumerable set.

Set A is denumerable iff A is equipollent to $\mathbb{N}$.

e) State the definition of a countable set.

Set A is countable iff A is equipollent to some subset of $\mathbb{N}$.

Excellent!

2. a) Let f and g be bounded functions, both with domain D. Then $f + g$ is a bounded function.

Well, since f is bounded, $\exists M_f \in \mathbb{R}$ so that $\forall x \in D$, $|f(x)| < M_f$, and since g is bounded, $\exists M_g \in \mathbb{R}$ so that $\forall x \in D$, $|g(x)| < M_g$.

Then by the Comparison Addition Principle $|f(x)| + |g(x)| < M_f + M_g$ must also hold for all $x \in D$, and by the Triangle Inequality we know $|f(x) + g(x)| < |f(x)| + |g(x)|$, so we have

$$|f(x) + g(x)| \quad |f(x)| + |g(x)| < M_f + M_g$$

But then by the Transitive Property we can conclude

$$|f(x) + g(x)| < M_f + M_g$$

holds for all $x \in D$, where $M_f + M_g$ is a real number, so $f+g$ is bounded. $\square$

b) Let $m \in \mathbb{N}$, and let f_i be a bounded function from D to $\mathbb{R}$ for each $i \in \{n \in \mathbb{N} \mid n \leq m\}$.

Then $\sum_{i=0}^{m} f_i$ is a bounded function.

Let's induct!

When $m = 0$, $\sum_{i=0}^{0} f_i = 0$, which is certainly bounded, so our base case holds.

Now s'pose $\sum_{i=0}^{k} f_i$ is bounded, and consider $\sum_{i=0}^{k+1} f_i$.

Well, $\sum_{i=0}^{k+1} f_i = \sum_{i=0}^{k} f_i + f_{k+1}$. We know f_{k+1} is bounded, and $\sum_{i=0}^{k} f_i$ is bounded by our inductive hypothesis. Then the sum of these two bounded functions is bounded by part a. Thus if the statement holds for k, it must also hold for $k+1$

Then by mathematical induction the statement holds for all $m \in \mathbb{N}$. $\square$

3. If f:$A \rightarrow B$ and g:$B \rightarrow C$ are injective functions, then $g \circ f$ is injective.

Let $f: A \to B$ and $g: B \to C$ be injective functions. Suppose $(g \circ f)(a_1) = (g \circ f)(a_2)$ for $a_1, a_2 \in A$. Then by definition of composition, $g(f(a_1)) = g(f(a_2))$. Since g is injective, by definition, $f(a_1) = f(a_2)$. Since f is injective, $a_1 = a_2$. Therefore, we can conclude that $g \circ f$ is injective. $\square$

Excellent!

4. In class we used the fact that $f(n) = \frac{n-1}{2}$ is a bijection from the odd naturals to the naturals. Prove that it is.

We need to show f is injective and surjective.

Injective: S'pose $f(a) = f(b)$ for some $a, b \in Odds$.

Then $\frac{a-1}{2} = \frac{b-1}{2}$

So $a - 1 = b - 1$

And $a = b$, so $f(a) = f(b) \Rightarrow a = b$ and f is injective.

Surjective: Take $a \in \mathbb{N}$ as an arbitrary element of the codomain.

But $2a+1$ is in the odd naturals, our domain.

And $f(2a+1) = \frac{(2a+1)-1}{2} = \frac{2a}{2} = a$. So for any element of our codomain, we have an element of our domain that f sends to it, and f is surjective.

Thus f is both injective and surjective, and therefore a bijection.

Scratch:

Take $a \in \mathbb{N}$

$a = \frac{n-1}{2}$

$2a = n-1$

$2a+1 = n$

5. If A is equipollent to B, then B is equipollent to A.

Let set A be equipollent to B $\ni$ $f: A \to B$ is bijective, by a previous proof we know that there is a function g which is the inverse function and is bijective.

$\therefore$ there is a function $g: B \to A$ which is bijective

$\therefore$ by definition of equipollent, B is equipollent to A $\square$

Great