Examlet 4 Foundations of Advanced Math 4/18/14

1. a) State the definition of a relation from A to B.

a relation from A to B can be defined as a subset of A x B

b) State the definition of a partition of a set A.

a partition of a set A is a set of pairwise disjoint nonempty sets whose union is the set of A

c) State the definition of a graph.

a graph G is a set V of vertices along with a set E of edges, where each edge is a set of exactly two vertices

Excellent!

2. Consider the relation $\sim$ on $\mathbb{Z}$ defined by $a \sim b$ iff $a - b$ is threven. Show that $\sim$ is an equivalence relation, being clear about your reasoning.

We need to show $\sim$ is reflexive, symmetric, and transitive.

Reflexive: Take $a \in \mathbb{Z}$. Then $a - a = 0$, and 0 is threven since $0 = 3 \cdot 0$, with $0 \in \mathbb{Z}$. Thus $a \sim a$ as desired, and $\sim$ is reflexive.

Symmetric: S'pose $a \sim b$, so we know $a - b = 3n$ for $n \in \mathbb{Z}$. But then we also know $-n \in \mathbb{Z}$, and $b - a = 3(-n)$, so $b \sim a$, and thus $\sim$ is symmetric.

Transitive: S'pose $a \sim b$ and $b \sim c$, so we know $a - b = 3n$ for $n \in \mathbb{Z}$ and $b - c = 3m$ for $m \in \mathbb{Z}$. Adding these equations gives us $(a-b)+(b-c) = 3n + 3m$, or $a - c = 3(n+m)$, where $n + m \in \mathbb{Z}$ by C.o.I., so $a \sim c$ and thus $\sim$ is transitive.

So since $\sim$ is reflexive, symmetric, and transitive, it's an equivalence relation. $\square$

3. a) Express the definition of a surjective function formally in terms of ordered pairs.

$f: A \to B$ is surjective iff $\forall b \in B \ \exists a \in A \ni (a, b) \in f$.

b) Express the definition of an increasing function formally in terms of ordered pairs.

A function $f: D \to \mathbb{R}$ is <u>increasing</u> iff $\forall x, y \in D$ with $x < y$ and $(x, X) \in f \wedge (y, Y) \in f$, we also have $X \geq Y$.

4. a) Let S be a set and Π a partition of S defined by $a \sim b \Leftrightarrow \exists P \in \Pi$ for which $a, b \in P$.
Then $\sim$ is a reflexive relation.

Let $x \in S$. We know $\exists P \in \Pi$ such that $x \in P$, because by definition, a partition is a set of pairwise, nonempty subsets of S, whose union is all of S. So, since $x \in P$, $x \in P$, so $x \sim x$, and we can conclude that $\sim$ is reflexive. $\square$

Great

b) Let S be a set and Π a partition of S defined by $a \sim b \Leftrightarrow \exists P \in \Pi$ for which $a, b \in P$.
Then $\sim$ is a symmetric relation.

To prove $\sim$ is a symmetric relation, we must show that $x \sim y \Rightarrow y \sim x$. So, suppose $x \sim y$, so $\exists P \in \Pi$ such that $x, y \in P$. Since $x, y \in P$, $y, x \in P$, so $y \sim x$, so we can conclude that $\sim$ is symmetric. $\square$

Nice!

[redacted]et S be a set and Π a partition of S defined by $a \sim b \Leftrightarrow \exists P \in \Pi$ for which $a, b \in P$.
[redacted]hen $\sim$ is a transitive relation.

To prove $\sim$ is a transitive relation, we must show that $x \sim y \land y \sim z \Rightarrow x \sim z$.
So, suppose $x \sim y$ and $y \sim z$, so $\exists P \in \Pi$ such that $x, y \in P$ and $\exists Q \in \Pi$ such that $y, z \in Q$. Since a partition is made up of pairwise disjoint nonempty subsets, the fact that $y \in P$ and $y \in Q$ can only be possible if $P = Q$. So, since $y, z \in Q$, $y, z \in P$. Therefore, $x, y, z \in P$, so we know that $x \sim z$, and can conclude that $\sim$ is transitive. $\square$

Excellent

5. a) In any graph, the number of vertices of odd degree is even.

Well, s'pose instead that there are an odd number of vertices of odd degree. We proved the sum of an odd number of odds is odd, and the sum of any number of evens is even, and the sum of those odd and even subtotals is odd. But this contradicts our previous result that the total degree of vertices in a graph is even. So since s'posing an odd number of odd vertices is contradictory, it must instead be the case that the number of odd vertices is even. □

b) If a graph G is connected, then the graph G' having the same vertex set and an edge set with exactly one fewer element is also connected.

<u>Counterexample</u>: Let G be ,

but G' be