disk of radius a. Consider the axis perpendicular to the disk and through its center. Find the electric potential at the point P on this axis at a distance R from the center. (See Figure 8.56.)

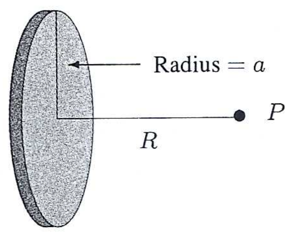


Figure 8.56

For Problems 23–24, find the kinetic energy of the rotating body. Use the fact that the kinetic energy of a particle of mass m moving at a speed v is $\frac{1}{2}mv^2$. Slice the object into pieces in such a way that the velocity is approximately constant on each piece.

23. Find the kinetic energy of a rod of mass 10 kg and length 6 m rotating about an axis perpendicular to the rod at its midpoint, with an angular velocity of 2 radians per second. (Imagine a helicopter blade of uniform thickness.)

24. Find the kinetic energy of a phonograph record of uniform density, mass 50 gm and radius 10 cm rotating at $33\frac{1}{3}$ revolutions per minute.

For Problems 25–27, find the gravitational force between two objects. Use the fact that the gravitational attraction between particles of mass m_1 and m_2 at a distance r apart is Gm_1m_2/r^2. Slice the objects into pieces, use this formula for the pieces, and sum using a definite integral.

25. What is the force of gravitational attraction between a thin uniform rod of mass M and length l and a particle of mass m lying in the same line as the rod at a distance a from one end?

26. Two long, thin, uniform rods of lengths l_1 and l_2 lie on a straight line with a gap between them of length a. Suppose their masses are M_1 and M_2, respectively, and the constant of the gravitation is G. What is the force of attraction between the rods? (Use the result of Problem 25.)

27. Find the gravitational force exerted by a thin uniform ring of mass M and radius a on a particle of mass m lying on a line perpendicular to the ring through its center. Assume m is at a distance y from the center of the ring.

28. A uniform, thin, circular disk of radius a and mass M lies on a horizontal plane. The point P lies a distance y directly above O, the center of the disk. Calculate the gravitational force on a mass m at the point P. (See Figure 8.57.) Use the fact that the gravitational force exerted on the mass m by a thin horizontal ring of radius r, mass μ, and center O is toward O and given by

$$F = \frac{G\mu m y}{(r^2 + y^2)^{3/2}}, \quad \text{where } G \text{ is constant.}$$

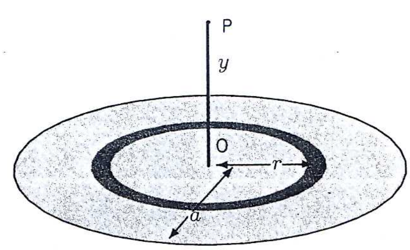


Figure 8.57

8.5 APPLICATIONS TO ECONOMICS

Present and Future Value

Many business deals involve payments in the future. If you buy a car or furniture, for example, you may buy it on credit and pay over a period of time. If you are going to accept payment in the future under such a deal, you obviously need to know how much you should be paid. Being paid \$100 in the future is clearly worse than being paid \$100 today for many reasons. If you are given the money today, you can do something else with it—for example, put it in the bank, invest it somewhere, or spend it. Thus, even without considering inflation, if you are to accept payment in the future, you would expect to be paid more to compensate for this loss of potential earnings. The question we will consider now is, how much more?

To simplify matters, we consider only what we would lose by not earning interest; we will not consider the effect of inflation. Let's look at some specific numbers. Suppose you deposit \$100 in an account which earns 7% interest compounded annually, so that in a year's time you will have \$107. Thus, \$100 today will be worth \$107 a year from now. We say that the \$107 is the *future value* of the \$100, and that the \$100 is the *present value* of the \$107. Observe that the present value is smaller than the future value. In general, we say the following:

- The **future value**, $\$B$, of a payment, $\$P$, is the amount to which the $\$P$ would have grown if deposited in an interest bearing bank account.
- The **present value**, $\$P$, of a future payment, $\$B$, is the amount which would have to be deposited in a bank account today to produce exactly $\$B$ in the account at the relevant time in the future.

With an interest rate of r, compounded annually, and a time period of t years, a deposit of $\$P$ grows to a future balance of $\$B$, where

$$B = P(1+r)^t, \quad \text{or equivalently,} \quad P = \frac{B}{(1+r)^t}.$$

Note that for a 7% interest rate, $r = 0.07$. If instead of annual compounding, we have continuous compounding, we get the following result:

$$B = Pe^{rt}, \quad \text{or equivalently,} \quad P = \frac{B}{e^{rt}} = Be^{-rt}.$$

Example 1 You win the lottery and are offered the choice between $1 million in four yearly installments of $250,000 each, starting now, and a lump-sum payment of $920,000 now. Assuming a 6% interest rate, compounded continuously, and ignoring taxes, which should you choose?

Solution We will do the problem in two ways. First, we assume that you pick the option with the largest present value. The first of the four $250,000 payments is made now, so

$$\text{Present value of first payment} = \$250{,}000.$$

The second payment is made one year from now, so

$$\text{Present value of second payment} = \$250{,}000e^{-0.06(1)}.$$

Calculating the present value of the third and fourth payments similarly, we find:

$$\begin{aligned}\text{Total present value} &= \$250{,}000 + \$250{,}000e^{-0.06(1)} + \$250{,}000e^{-0.06(2)} + \$250{,}000e^{-0.06(3)}\\ &\approx \$250{,}000 + \$235{,}441 + \$221{,}730 + \$208{,}818\\ &= \$915{,}989.\end{aligned}$$

Since the present value of the four payments is less than $920,000, you are better off taking the $920,000 right now.

Alternatively, we can compare the future values of the two pay schemes. The scheme with the highest future value is the best from a purely financial point of view. We calculate the future value of both schemes three years from now, on the date of the last $250,000 payment. At that time,

$$\text{Future value of the lump sum payment} = \$920{,}000e^{0.06(3)} \approx \$1{,}101{,}440.$$

Now we calculate the future value of the first $250,000 payment:

$$\text{Future value of the first payment} = \$250{,}000e^{0.06(3)}.$$

Calculating the future value of the other payments similarly, we find:

$$\begin{aligned}\text{Total future value} &= \$250{,}000e^{0.06(3)} + \$250{,}000e^{0.06(2)} + \$250{,}000e^{0.06(1)} + \$250{,}000\\ &\approx \$299{,}304 + \$281{,}874 + \$265{,}459 + \$250{,}000\\ &= \$1{,}096{,}637.\end{aligned}$$

The future value of the $920,000 payment is greater, so you are better off taking the $920,000 right now. Of course, since the present value of the $920,000 payment is greater than the present value of the four separate payments, you would expect the future value of the $920,000 payment to be greater than the future value of the four separate payments.
(Note: If you read the fine print, you will find that many lotteries do not make their payments right away, but often spread them out, sometimes far into the future. This is to reduce the present value of the payments made, so that the value of the prizes is much less than it might first appear!)

Income Stream

When we consider payments made to or by an individual, we usually think of *discrete* payments, that is, payments made at specific moments in time. However, we may think of payments made by a company as being *continuous*. The revenues earned by a huge corporation, for example, come in essentially all the time and can be represented by a continuous *income stream*, written

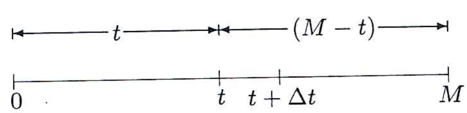
$$P(t) \text{ dollars/year.}$$

Notice that $P(t)$ is the *rate* at which deposits are made (its units are dollars per year, for example) and that this rate may vary with time, t.

Present and Future Values of an Income Stream

Just as we can find the present and future values of a single payment, so we can find the present and future values of a stream of payments. We will assume that interest is compounded continuously.

Suppose that we want to calculate the present value of the income stream described by a rate of $P(t)$ dollars per year, and that we are interested in the period from now until M years in the future. We divide the stream into many small deposits, each of which is made at approximately one instant. We divide the interval $0 \le t \le M$ into subintervals, each of length Δt:



Assuming Δt is small, the rate, $P(t)$, at which deposits are being made will not vary much within one subinterval. Thus, between t and $t + \Delta t$:

$$\begin{aligned} \text{Amount deposited} &\approx \text{Rate of deposits} \times \text{Time} \\ &\approx (P(t) \text{ dollars/year})(\Delta t \text{ years}) \\ &= P(t)\Delta t \text{ dollars.} \end{aligned}$$

Measured from the present, the deposit of $P(t)\Delta t$ is made t years in the future. Thus,

$$\begin{matrix}\text{Present value of money deposited} \\ \text{in interval } t \text{ to } t+\Delta t\end{matrix} \approx P(t)\Delta t e^{-rt}.$$

Summing over all subintervals gives

$$\text{Total present value} \approx \sum P(t)e^{-rt}\Delta t \text{ dollars.}$$

In the limit as $\Delta t \to 0$, we get the following integral:

$$\boxed{\text{Present value} = \int_0^M P(t)e^{-rt}dt \text{ dollars.}}$$

In computing future value, the deposit of $P(t)\Delta t$ has a period of $(M - t)$ years to earn interest, and therefore

$$\begin{matrix}\text{Future value of money deposited} \\ \text{in interval } t \text{ to } t+\Delta t\end{matrix} \approx [P(t)\Delta t]\, e^{r(M-t)}.$$

Summing over all subintervals, we get:

$$\text{Total future value} \approx \sum P(t)\Delta t e^{r(M-t)} \text{ dollars.}$$

As the length of the subdivisions tends toward zero, the sum becomes an integral:

$$\text{Future value} = \int_0^M P(t)e^{r(M-t)}\,dt \text{ dollars.}$$

In addition, by writing $e^{r(M-t)} = e^{rM} \cdot e^{-rt}$ and factoring out e^{rM}, we see that

$$\text{Future value} = e^{rM} \cdot \text{Present value.}$$

Example 2 Find the present and future values of a constant income stream of \$1000 per year over a period of 20 years, assuming an interest rate of 10% compounded continuously.

Solution Using $P(t) = 1000$ and $r = 0.1$, we have

$$\text{Present value} = \int_0^{20} 1000e^{-0.1t}dt = 1000\left(-\frac{e^{-0.1t}}{0.1}\right)\Bigg|_0^{20} = 10{,}000(1-e^{-2}) \approx 8646.65 \text{ dollars.}$$

There are two ways to compute the future value. Using the present value of \$8646.65, we have

$$\text{Future value} = 8646.65e^{0.1(20)} = 63{,}890.58 \text{ dollars.}$$

Alternatively, we can use the integral formula:

$$\begin{aligned}\text{Future value} &= \int_0^{20} 1000e^{0.1(20-t)}dt = \int_0^{20} 1000e^2e^{-0.1t}dt \\ &= 1000e^2\left(-\frac{e^{-0.1t}}{0.1}\right)\Bigg|_0^{20} = 10{,}000e^2(1-e^{-2}) \approx 63890.58 \text{ dollars.}\end{aligned}$$

Notice that the total amount deposited is \$1000 per year for 20 years, or \$20,000. The additional \$43,895.58 of the future value comes from interest earned.

Supply and Demand Curves

In a free market, the quantity of a certain item produced and sold can be described by the supply and demand curves of the item. The *supply curve* shows the quantity of the item the producers will supply at different price levels. It is usually assumed that as the price increases, the quantity supplied will increase. The consumers' behavior is reflected in the *demand curve*, which shows what quantity of goods are bought at various prices. An increase in price is usually assumed to cause a decrease in the quantity purchased. See Figure 8.58.

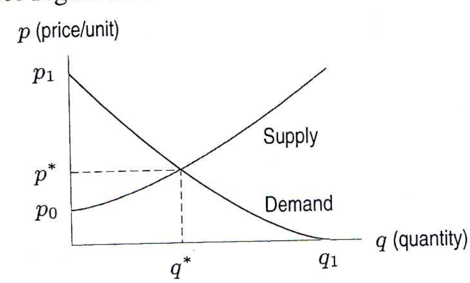


Figure 8.58: Supply and demand curves

It is assumed that the market settles to the *equilibrium price* and *quantity*, p^* and q^*, where the graphs cross. At equilibrium, a quantity q^* of an item is produced and sold for a price of p^* each.

See Figure 8.60. Similarly, if the supply curve is given by the function $p = S(q)$ and p^* and q^* are equilibrium price and quantity, the producer surplus is calculated as follows:

$$\text{Producer surplus} = p^*q^* - \left(\int_0^{q^*} S(q)dq\right) = \begin{array}{c}\text{Area between supply}\\ \text{curve and line } p = p^*.\end{array}$$

Exercises and Problems for Section 8.5

Exercises

1. Find the future value of an income stream of \$1000 per year, deposited into an account paying 8% interest, compounded continuously, over a 10-year period.
2. Find the present and future values of an income stream of \$3000 per year over a 15-year period, assuming a 6% annual interest rate compounded continuously.
3. Find the present and future values of an income stream of \$2000 a year, for a period of 5 years, if the continuous interest rate is 8%.
4. A person deposits money into a retirement account, which pays 7% interest compounded continuously, at a rate of \$1000 per year for 20 years. Calculate:
   (a) The balance in the account at the end of the 20 years.
   (b) The amount of money actually deposited into the account.
   (c) The interest earned during the 20 years.

Problems

5. Draw a graph, with time in years on the horizontal axis, of what an income stream might look like for a company that sells sunscreen in the northeast United States.
6. On April 15, 1999, Maria Grasso won the largest lottery amount ever awarded up to that date. She was given her choice between \$197 million, paid out continuously over 26 years, or a lump sum of \$104 million, paid immediately.
   (a) Which option is better if the interest rate is 6%, compounded continuously ? An interest rate of 5%?
   (b) The winner chose the lump sum option. What assumption was she making about interest rates?
7. (a) A bank account earns 10% interest compounded continuously. At what (constant, continuous) rate must a parent deposit money into such an account in order to save \$100,000 in 10 years for a child's college expenses?
   (b) If the parent decides instead to deposit a lump sum now in order to attain the goal of \$100,000 in 10 years, how much must be deposited now?
8. (a) If you deposit money continuously at a constant rate of \$1000 per year into a bank account that earns 5% interest, how many years will it take for the balance to reach \$10,000?
   (b) How many years would it take if the account had \$2000 in it initially?
9. A business associate who owes you \$3000 offers to pay you \$2800 now, or else pay you three yearly installments of \$1000 each, with the first installment paid now. If you use only financial reasons to make your decision, which option should you choose? Justify your answer, assuming a 6% market interest rate, compounded continuously.
10. Big Tree McGee is negotiating his rookie contract with a professional basketball team. They have agreed to a three-year deal which will pay Big Tree a fixed amount at the end of each of the three years, plus a signing bonus at the beginning of his first year. They are still haggling about the amounts and Big Tree must decide between a big signing bonus and fixed payments per year, or a smaller bonus with payments increasing each year. The two options are summarized in the table. All values are payments in millions of dollars.

	Signing bonus	Year 1	Year 2	Year 3
Option #1	6.0	2.0	2.0	2.0
Option #2	1.0	2.0	4.0	6.0

   (a) Big Tree decides to invest all income in stock funds which he expects to grow at a rate of 10% per year, compounded continuously. He would like to choose the contract option which gives him the greater future value at the end of the three years when the last payment is made. Which option should he choose?
   (b) Calculate the present value of each contract offer.
11. Sales of Version 6.0 of a computer software package start out high and decrease exponentially. At time t, in years, the sales are $s(t) = 50e^{-t}$ thousands of dollars per

137 Does not converge

139 Does not converge

141 5/6

143 11/3

145 (a) (i) 0
(ii) $\frac{2}{\pi}$
(iii) $\frac{1}{2}$
(b) Smallest to largest:
Average value of $f(t)$
Average value of $k(t)$
Average value of $g(t)$

147 error for TRAP(10) ≈ 0.0078

149 (a) 0.5 ml
(b) 99.95%

151 (a) $(\ln x)^2/2, (\ln x)^3/3, (\ln x)^4/4$
(b) $(\ln x)^{n+1}/(n+1)$

153 (a) $(-9\cos x + \cos(3x))/12$
(b) $(3\sin x - \sin(3x))/4$

155 (a) $x + x/(2(1+x^2)) - (3/2)\arctan x$
(b) $1 - (x^2/(1+x^2)^2) - 1/(1+x^2)$

Ch. 7 Understanding

1 False

3 False

5 True

7 False

9 False

11 False

13 True

15 False

17 True

19 True

21 False

23 False

25 True

27 False.

Section 8.1

1 15

3 15/2

5 $(5/2)\pi$

7 1/6

9 $\int_0^9 4\pi\,dx = 36\pi$ cm^3

11 $\int_0^5 (4\pi/25)y^2\,dy = 20\pi/3$ cm^3

13 $\int_0^5 \pi(5^2 - y^2)\,dy = 250\pi/3$ mm^3

15 5 to 1

17 Triangle; $b, h = 1, 3$

19 Quarter circle $r = \sqrt{15}$

21 Hemisphere, $r = 12$

23 Cone, $h = 6, r = 3$

25 $V = (4\pi r^3)/3$

27 (a) $3\Delta x$; $\int_0^4 3\,dx = 12$ cm^3
(b) $8(1 - h/3)\Delta h$; $\int_0^3 8(1 - h/3)\,dh = 12$ cm^3

29 $\int_0^{150} 1400(160 - h)\,dh = 1.785 \cdot 10^7$ m^3

Section 8.2

1 $\pi/5$

3 $\pi(e^2 - e^{-2})/2$

5 $256\pi/15$

7 $\pi^2/4$

9 $\pi((e^6/6) - (e^2/2) + 1/3)$

11 3.526

13 $e - 1$

15 (a) $16\pi/3$
(b) 1.48

17 $V = (16/7)\pi \approx 7.18$

19 $V = (\pi^2/2) \approx 4.935$

21 $V \approx 42.42$

23 $V = (e^2 - 1/2) \approx 3.195$

25 (a) Volume ≈ 152 in^3
(b) About 15 apples

27 $40,000LH^{3/2}/(3\sqrt{a})$

29 (a) $dh/dt = -6/\pi$
(b) $t = \pi/6$

31 (a) $4\int_0^r \sqrt{1 + (-\frac{x}{y})^2}\,dx$
(b) $2\pi r$

33 $e - e^{-1}$

Section 8.3

1 $1 - e^{-10}$ gm

3 (a) $\sum_{i=1}^{N}(2 + 6x_i)\Delta x$
(b) 16 grams

5 (b) $\sum_{i=1}^{N}[600 + 300\sin(4\sqrt{x_i + 0.15})]\frac{20}{N}$
(c) ≈ 11513

7 2 cm to right of origin

9 1 gm

11 (a) $\int_0^5 2\pi r(0.115e^{-2r})dr$
(b) 181 cubic meters

13 (a) $\pi r^2 l/2$
(b) $2klr^3/3$

15 $\int_0^{60} \frac{1}{144}g(t)\,dt$ ft^3

17 $x = 2$

19 $\pi/2$

21 (a) Right
(b) $2/(1 + 6e - e^2) \approx 0.2$

23 (a) 10/3 gm
(b) $\bar{x} = 3/5$ cm; $\bar{y} = 3/8$ cm

25 1.25 cm from center of base

27 (a) $16000\delta/3$ gm
(b) 2.5 cm above center of base

Section 8.4

1 9/2 joules

3 (a) 1.5 joules, 13.5 joules
(b) For $x = 4$ to $x = 5$ Force larger

5 $1.489 \cdot 10^{10}$ joules

7 11,000 ft-lbs

9 1,404,000 ft-lb

11 2,822,909.50 ft-lb

13 354,673 ft-lb

15 (a) Force on dam $\approx \sum_{i=0}^{N-1} 1000(62.4h_i)\Delta h$
(b) $\int_0^{50} = 1000(62.4h)\,dh =$ 78,000,000 pounds

17 Bottom: 1497.6 lbs
Front and back: 499.2 lbs
Both sides: 374.4 lbs

19 (a) 21,840 lb/ft^2; 151.7 lb/in^2
(b) (i) 546,000 pounds
(ii) 542,100 pounds

21 $9800\int_0^{100} h(3600 - 6h)\,dh = 1.6 \cdot 10^{11}$ newtons

23 60 joules

25 $(GmM)/(a(a + l))$

27 $GMmy/(a^2 + y^2)^{3/2}$ toward center

Section 8.5

1 \$15,319.30.

3 \$8, 242, \$12, 296

7 (a) \$5820 per year
(b) \$36,787.94

9 Installments

11 \$46,800

13 (a) 10.6 years
(b) 624.9 million dollars

15 \$85,750,000

19 (a) Less
(b) Can't tell
(c) Less

Section 8.6

5 pdf; 1/2

7 pdf; 2/3

9 pdf; 2

11 (a) 0.9 m–1.1 m

15 (a) Cumulative distribution increasing
(b) Vertical 0.2, horizontal 2

17 (a) 22.1%
(b) 33.0%
(c) 30.1%
(d) $C(h) = 1 - e^{-0.4h}$

19 (b) About 3/4

Section 8.7

5 (a) 0.684 : 1
(b) 1.6 hours
(c) 1.682 hours

7 (a) $P(t)$ = Fraction of population who survive up to t years after treatment
(b) $S(t) = e^{-Ct}$
(c) 0.178

9 (a) $p(x) = \left(e^{-\frac{1}{2}\left(\frac{x-100}{15}\right)^2}\right) / \left(15\sqrt{2\pi}\right)$
(b) 6.7% of the population

11 (c) μ represents the mean of the distribution, while σ is the standard deviation.

13 (b)

15 (a) $p(r) = 4r^2e^{-2r}$
(b) Mean: 1.5 Bohr radii
Median: 1.33 Bohr radii
Most likely: 1 Bohr radius