Exam 3 Calc 2 4/1/2016

Each problem is worth 10 points. For full credit provide complete justification for your answers.

1. Determine the value of $\sum_{n=1}^{\infty} \frac{1}{3^n}$.

$\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^{-n}$ $a = \frac{1}{3}$ $r = \frac{1}{3}$

$\frac{\frac{1}{3}}{1 - 1/3} = \frac{\frac{1}{3}}{\frac{2}{3}} = \frac{1}{\cancel{3}} \cdot \frac{\cancel{3}}{2} = \boxed{\frac{1}{2}}$

Geometric Series!

Great!

2. Determine whether $\sum_{n=1}^{\infty} \frac{n}{n^3+3}$ converges or diverges.

The Comparison test

lets take $\frac{n}{n^3} > \frac{n}{n^3+3}$

then $\frac{n}{n^3} = \frac{1}{n^2}$. (P series test)

because the power on n is >1

we know that $\frac{1}{n^2}$ converges.

Since $\frac{n}{n^3+3}$ is smaller than $\frac{n}{n^3}$ and we know $\frac{n}{n^3}$ converges, that would also mean that $\frac{n}{n^3+3}$ is convergent.

Great

3. Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges or diverges.

Alternating Series test

- Alternating $(-1)^n$ ✓
- rest of it is positive $\frac{1}{\sqrt{n}}$ ✓
- $\lim_{n \to \infty} \frac{1}{\sqrt{n}} = 0$ ✓
- decreasing $\frac{d\, n^{-\frac{1}{2}}}{dn} = -\frac{1}{2} n^{-\frac{3}{2}}$

the derivative is negative so it is decreasing ✓

Excellent!

$\therefore \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges by AST

4. Set up an integral and evaluate it to find the arc length of $y = x^{3/2}$ on the interval [1,2].

Arc Length $= \int_a^b \sqrt{1 + f'(x)^2}\, dx$

$f(x) = x^{3/2}$

$f'(x) = \frac{3}{2} x^{1/2}$

$= \int_1^2 \sqrt{1 + [3/2(x^{1/2})]^2}\, dx$

$= \int_1^2 \sqrt{1 + 9/4 x}\, dx$

let $u = 1 + 9/4 x$

$du = \frac{9}{4} dx$

$dx = \frac{4}{9} du$

$= \frac{4}{9} \int u^{1/2}\, du$

$= \frac{4}{9}\left(\frac{2}{3} u^{3/2}\right)$

$= \frac{8}{27} u^{3/2}$

Good

$= \frac{8}{27}\left(1 + 9/4 x\right)^{3/2} \Big|_1^2$

$= \frac{8}{27}\left(1 + \frac{9}{4}(2)\right)^{3/2} - \frac{8}{27}\left(1 + \frac{9}{4}\right)^{3/2}$

$= \frac{8}{27}\left(\frac{11}{2}\right)^{3/2} - \frac{8}{27}\left(\frac{13}{4}\right)^{3/2} \approx 2.0858$

5. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^3}$ converges or diverges.

use integral test

$$= \lim_{b\to\infty} \int_2^b \frac{1}{x(\ln x)^3}\,dx$$

let $u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \lim_{b\to\infty} \int \frac{1}{u^3} \cdot \frac{1}{x}\,dx$$

$$= \lim_{b\to\infty} \int u^{-3}\,du$$

$$= \lim_{b\to\infty} \frac{-1}{2u^2} \rightarrow = \lim_{b\to\infty} \left.\frac{-1}{2(\ln x)^2}\right|_2^b \rightarrow = \lim_{b\to\infty} \frac{-1}{2(\ln b)^2} - \frac{-1}{2(\ln 2)^2}$$

$$= 0 + \frac{1}{2(\ln 2)^2}$$

$$= \frac{1}{2(\ln 2)^2}$$

since the integral converges the series also converges

Excellent!

6. Find the Taylor series for $f(x) = \sin x$ centered at $x = \pi/2$.

$$\frac{f^{(n)}(c)\cdot(x-c)^n}{n!}$$

$$T(x) = \frac{1\left(x-\frac{\pi}{2}\right)^0}{0!} + 0 - \frac{1\left(x-\frac{\pi}{2}\right)^2}{2!} + 0 + \frac{1\left(x-\frac{\pi}{2}\right)^4}{4!}$$

$f(x) = \sin x \qquad f\left(\frac{\pi}{2}\right) = 1$

$f'(x) = \cos x \qquad f'\left(\frac{\pi}{2}\right) = 0$

$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{2}\right) = -1$

$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{2}\right) = 0$

$f^{(4)}(x) = \sin x \qquad f^{(4)}\left(\frac{\pi}{2}\right) = 1$

$$\sum_{n=0}^{\infty} \frac{(-1)^n\left(x-\frac{\pi}{2}\right)^{2n}}{(2n)!}$$

Great!

7. Biff is a calculus student at Enormous State University, and he's having some trouble. Biff says "Well, crap. This series stuff is hard. I don't even know what half of it means. What the heck is the difference between absolute convergence and conditional convergence anyway? I mean, either it converges or not, right?"

Help Biff by explaining clearly the difference between conditional and absolute convergence.

Biff, an example is the best way to explain this lets say we have $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, this converges because of the A.S.T., however this is an example of conditional convergence because if we took the absolute value of $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right|$ we would get $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, which is a p-series with $p = \frac{1}{2}$, if the $p \leq 1$ it diverges. Wow, look at that. the series $\sum \frac{(-1)^n}{\sqrt{n}}$ converges, but the absolute value, $\sum \frac{1}{\sqrt{n}}$ diverges. This must mean the series does not converge absolutely. The absolute value of the series does not also converge, so we cannot say it does so absolutely.

Excellent!

8. Use a Taylor series with at least 3 nonzero terms to approximate ln 0.9.

I know $\ln|1+x| = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

so $\ln|1-.1| = -1^0 \cdot \frac{-.1^1}{1} - 1 \cdot \frac{-.1^2}{2} + 1 \frac{-.1^3}{3}$

$\approx -.1 + \frac{.01}{2} - \frac{.001}{3}$

$\approx -.1 - .005 - .000333$

$\approx -.10533$ well done

Caclulator Says

$\ln .9 \approx -0.10536$

9. Determine the interval of convergence of the series $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

Ratio test

$$\lim_{n\to\infty} \left| \frac{\frac{x^{2(n+1)+1}}{(2(n+1)+1)!}}{\frac{x^{2n+1}}{(2n+1)!}} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^{2n+2+1}}{x^{2n+1}} \frac{(2n+1)!}{(2n+2+1)!} \right| \quad (2n+3)!$$

$$\lim_{n\to\infty} \left| \frac{x^2 (2n+1)!}{(2n+2)(2n+3)(2n+1)!} \right|$$

$$\lim_{n\to\infty} \left| \frac{x^2}{(2n+2)(2n+3)} \right|$$

$$\lim_{n\to\infty} \frac{1}{(2n+2)(2n+3)} |x^2|$$

$$= 0 |x^2|$$

$0 < 1$ for all x by Ratio test <1 converges

$\therefore$ interval of convergence is $(-\infty, \infty)$ for $\sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$

Excellent!

10. Determine the interval of convergence of the series $\sum_{n=1}^{\infty} n(x-4)^n$.

ratio test!

$$\lim_{n\to\infty} \left|\frac{(n+1)(x-4)^{n+1}}{n(x-4)^n}\right| = 1 \cdot \left|(x-4)\right|$$

so $\left|(x-4)\right| < 1$

so $-1 < x-4 < 1$

$3 < x < 5$

So the interval of converge

is $x \in (3,5)$

because of
Geometric test

$x \neq 3$

$x \neq 5$.

Outstanding!