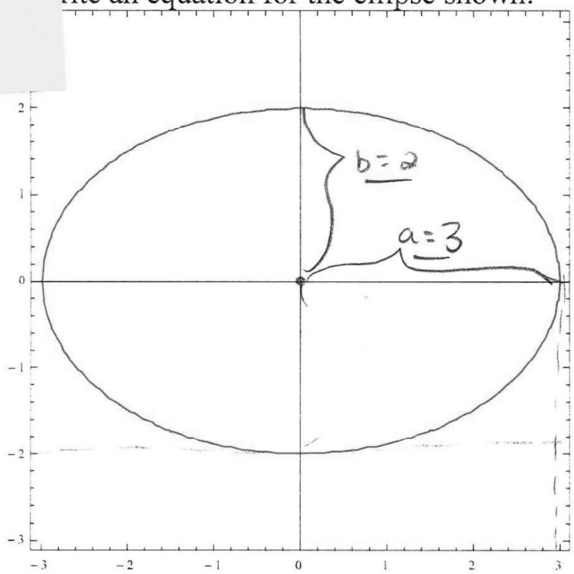
Exam 3 Calc 2 4/22/2016

Each problem is worth 10 points. For full credit provide complete justification for your answers.

rite an equation for the ellipse shown:



$(h,k) : (0,0)$

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

$$\boxed{\frac{x^2}{9}+\frac{y^2}{4}=1}$$

Great!

2. a) Convert the point with rectangular coordinates (5,5) into polar form.

$r=\sqrt{a^2+b^2}$ $\quad \theta = \arctan \frac{y}{x}$ $\quad \sqrt{25+25}=\sqrt{50}=5\sqrt{2}$

$\arctan(1) \approx .7854 = \frac{\pi}{4}$ $\quad (5\sqrt{2}, \frac{\pi}{4})$

b) Convert the point with rectangular coordinates (0,3) into polar form.

$\sqrt{9+0}=\sqrt{9}=3$

$(3, \frac{\pi}{2})$

c) Convert the point with rectangular coordinates (0,3) into a different polar form than you gave in part b.

$(3, \frac{-3\pi}{2})$

Excellent

d) Convert the point with polar coordinates $(2,\pi/2)$ into rectangular form.

$x = r\cos\theta = 2\cos\frac{\pi}{2} = 0$

$y = r\sin\theta = 2\sin\frac{\pi}{2} = 2$

$(0,2)$

e) Convert the point with polar coordinates $(1,3\pi/4)$ into rectangular form.

$x = \cos\frac{3\pi}{4} = -\frac{\sqrt{2}}{2}$

$y = \sin\frac{3\pi}{4} = \frac{\sqrt{2}}{2}$

$\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

3. Use separation of variables to find a general solution to the differential equation $2y' + 5y = 4$.

$$2y' + 5y = 4$$

$$2\frac{dy}{dx} + 5y = 4$$

$$2\frac{dy}{dx} = 4 - 5y$$

$$2dy = (4-5y)\,dx$$

$$\int \frac{1}{4-5y}\,dy = \int \frac{1}{2}\,dx$$

$u = 4-5y$

$\frac{du}{dy} = -5$

$\frac{du}{-5} = dy$

$$-\frac{1}{5}\int \frac{1}{u}\,du = \frac{1}{2}\int dx$$

$$-\frac{1}{5}\ln|u| + C = \frac{1}{2}x + C$$

C on both sides can be consolidated into one constant

$$-\frac{1}{5}\ln|4-5y| = \frac{1}{2}x + C$$

$$\ln|4-5y| = -\frac{5}{2}x + C$$

$$e^{(-\frac{5}{2}x + C)} = |4-5y|$$

e^x is always positive

$$e^{-\frac{5}{2}x} \cdot e^C = 4-5y$$

e^C is a constant

$$Ce^{-\frac{5}{2}x} = 4-5y$$

$$Ce^{-\frac{5}{2}x} - 4 = -5y$$

$C/-5$ is still a constant

$$\underline{Ce^{-\frac{5}{2}x} + \frac{4}{5} = y}$$

Great

4. Consider the parametric curve $c(t) = (t^3 + t, t^2 - 1)$. Calculate dy/dx at the point where $t = 3$.

$c(t) = (t^3+t, t^2-1)$

$x(t) = t^3+t \quad x'(t) = 3t^2+1$

$y(t) = t^2-1 \quad y'(t) = 2t$

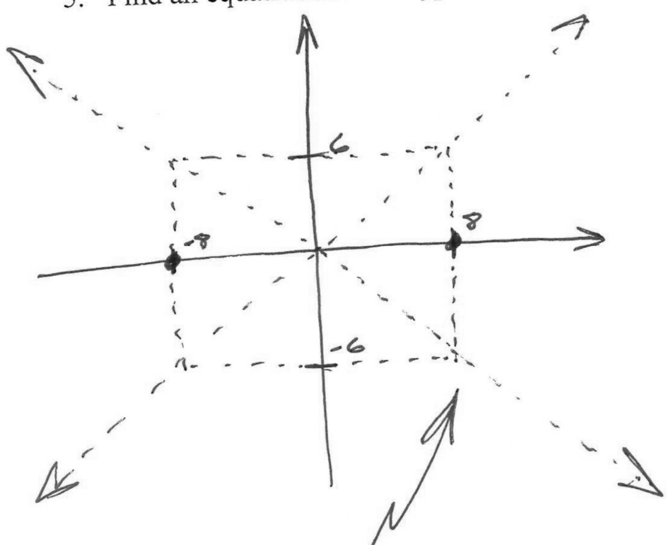
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2+1}$$

where $t=3$

$$\frac{dy}{dx} = \frac{2(3)}{3(3)^2+1} = \frac{6}{28} = \boxed{\frac{3}{14}}$$

Great

5. Find an equation for the hyperbola with vertices (±8,0) and asymptotes $y = \pm \frac{3}{4} x$.

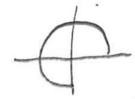


$$\frac{x^2}{8^2} - \frac{y^2}{6^2} = 1$$

If these lines have slope $\pm \frac{3}{4}$, then the other dimension of my half-imaginary rectangle satisfies $\frac{3}{4} = \frac{y}{8}$ so $y = 6$.

6. Biff is a calculus student at Enormous State University, and he has a question. Biff says "Dude, I love these parametric things, 'cause it's like all you gotta do is have your calculator graph 'em, you know? But for this one I think it screwed up somehow, 'cause it's $x = 3\cos t$ and $y = 3\sin t$, but the graph comes up like kind of a circle. That can't be right, 'cause trig stuff is all wavy, right?"

Help Biff by explaining what's going on.

Biff is correct by saying sin and cos are wavy, but in this case, each one determines only one part of the coordinate. 3cos t is oscillating between $x = 3$ and $x = -3$ at the same time and rate that 3 sin t oscillates between $y = 3$ and $y = -3$. Their oscillation results in the circle. X starts moving left as y moves up, making . At $\frac{\pi}{2}$ y moves down while x keeps going left to make . Y continues down; X switches right , and so on.

- Nice!

7. Consider the ellipse given by $c(t) = (3\cos t, 2\sin t)$. Set up an integral for the length of the first-quadrant portion of this curve.

$$\int_\alpha^\beta \sqrt{(x')^2 + (y')^2}\, dt$$

$$x = 0 = 3\cos t$$

$$0 = \cos t$$

$$\arcsin(0) = t$$

$$\frac{\pi}{2} = t$$

$$\int_0^{\pi/2} \sqrt{(-3\sin(t))^2 + (2\cos t)^2}\, dt$$

t value at which point function leave first quadrant

goes away

$$\int_0^{\pi/2} \sqrt{9\sin^2(t) + 4\cos^2(t)}\, dt$$

Excellent

8. Consider the ellipse given by $c(t) = (3\cos t, 2\sin t)$. Find the area of the region inside it.

$$A = \int_\alpha^\beta y(t) \cdot x'(t)\, dt$$

$$x = 3\cos(t) \qquad y = 2\sin(t)$$

$$x' = -3\sin(t)$$

Times 4 because only solving for Q1

$\frac{\pi}{2}$ to 0 instead of 0 to $\frac{\pi}{2}$ because the graph moves right to left

$$A = 4\int_{\pi/2}^{0} (2\sin(t))(-3\sin(t))\, dt$$

$$A = 4\int_{\pi/2}^{0} (-6\sin^2 t)\, dt$$

$$A = 12\left(\sin(t)\cos(t) - t\right)\Big|_{\pi/2}^{0} = 6\pi$$

Excellent!

9. Consider the family of parametric curves given by $x = at^2$ and $y = t^3 - 3t$. For which value(s) of a will the curve be perpendicular to itself at the point where it crosses itself?

Slope of tangent: $\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{3t^2 - 3}{2at}$

From calculator or setting $y=0$, I see the crossing point is when $t = \pm\sqrt{3}$, so

$$1 = \frac{3(\sqrt{3})^2 - 3}{2a(\sqrt{3})}$$

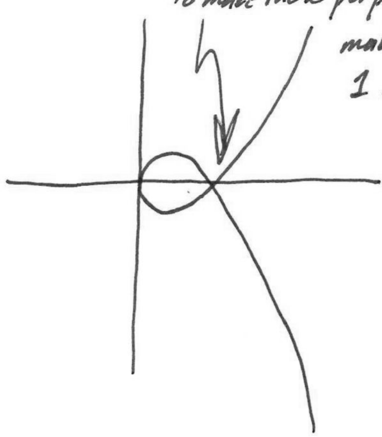
$$2\sqrt{3}\,a = 9 - 3$$

$$a = \frac{6}{2\sqrt{3}} = \frac{3}{\sqrt{3}} = \sqrt{3}$$

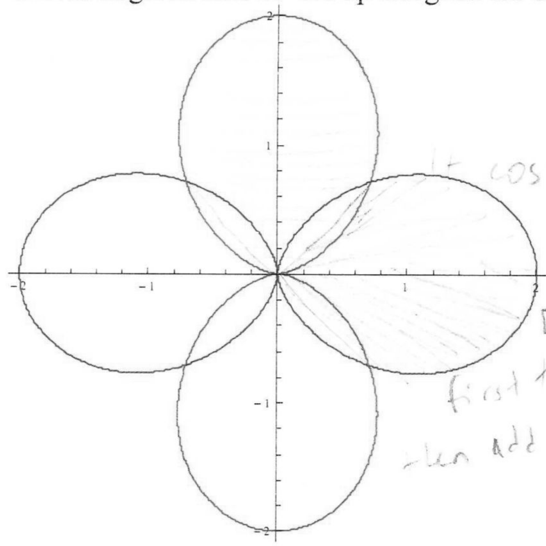
The flip will happen if I make that slope be -1, and give $a = -\sqrt{3}$, so

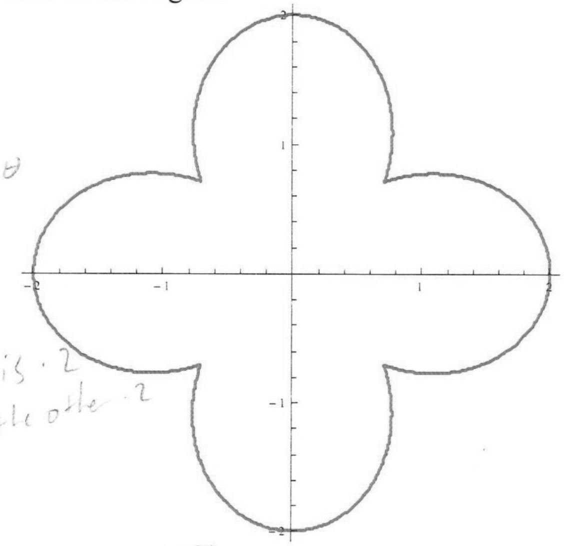
$$a = \pm\sqrt{3}$$

To make these perpendicular, make the slopes 1 and -1



10. For Earth Day, Jon is making a large green sign whose outline will be the region inside at least one of the curves $r = 1 + \cos 2\theta$ and $r = 1 - \cos 2\theta$ (with units in meters), so the region shown in green below. Set up integrals for the area of this region.





$1+\cos 2\theta = 1-\cos 2\theta$ intersect

$2\cos 2\theta = 0$

$\cos 2\theta = 0$

$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

$$2\cdot\frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} (1+\cos 2\theta)^2\, d\theta$$

$$+\,2\cdot\frac{1}{2}\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} (1-\cos 2\theta)^2\, d\theta$$

because of symmetry,

2 (Area of $1+\cos 2\theta$ from $-\frac{\pi}{4}$ to $\frac{\pi}{4}$)

+ 2 (Area of $1-\cos 2\theta$ from $\frac{\pi}{4}$ to $\frac{3\pi}{4}$) works

Yes it does!