Exam 1 Differential Equations 2/12/16

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. Find all values of c for which $y = \dfrac{1}{cx^2 + 1}$ is a solution to $y' + 16xy^2 = 0$.

$y = (cx^2+1)^{-1}$

$y' = \dfrac{-2cx}{(cx^2+1)^2}$

So $y' + 16xy^2 = 0 \Rightarrow \dfrac{-2cx}{(cx^2+1)^2} + \dfrac{16x}{(cx^2+1)^2} = 0$

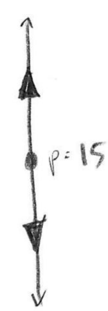
when $\underline{c = 8}$ Good

2. Sketch a phase line for the differential equation $\dfrac{dP}{dt} = \frac{1}{5}P - 3$. Clearly identify the equilibrium points as sources, sinks, or nodes.

$0 = \frac{1}{5}P - 3$

$3 = \frac{1}{5}P$

$15 = P$ is an equilibrium!



$P > 15$: $\dfrac{dP}{dt} > 0$

$P = 15$: $\dfrac{dP}{dt} = 0$

$P < 15$: $\dfrac{dP}{dt} < 0$

Great!

$P = 15$ is a $\underline{\text{source}}$.

3. Find a general solution to the differential equation $\frac{dP}{dt} = \frac{1}{5}P - 3$.

$$dP/dt = \frac{1}{5}(P-15)$$

$$\frac{dP}{P-15} = \frac{1}{5}\,dt$$

$$\ln|P-15| = \frac{1}{5}t + C$$

$$P - 15 = e^{\frac{1}{5}t + C} = Ae^{\frac{1}{5}t}$$

Constant A will take care of the absolute value.

$$\boxed{P = Ae^{\frac{1}{5}t} + 15}$$

Great

4. Consider the differential equation $\frac{dP}{dt} = \frac{1}{5}P - 3$, with initial condition $P(0) = 10$, modeling the population of parakeets (in thousands) living in a particular part of Paraguay. Use Euler's method with $\Delta t = 2$ to approximate the population after 4 years, when legislation (based on the Preserve Paraguay's Precious Pretty Parakeets Petition) is expected to alter the situation.

$P(0) = 10 \quad \Delta t = 2 \quad t_f = 4$

t	$P(t)$	$\frac{dP}{dt}$	ΔP
0	10	-1	-2
2	8	-1.4	-2.8
4	5.2		

After 4 years, $P(t) = 5.2$.

Good

5. Consider the differential equation $\dfrac{dy}{dt} = \dfrac{y^2 + ty}{y^2 + 3t^2}$. Change the dependent variable from y to u using the substitution $u = y/t$.

$$u = \frac{y}{t}, \qquad \frac{du}{dt} = \frac{dy}{dt} \cdot \frac{1}{t} - \frac{y}{t^2}$$

$$\frac{du}{dt} + \frac{y}{t^2} = \frac{dy}{dt} \cdot \frac{1}{t} \Rightarrow \underline{\frac{dy}{dt} = \frac{du}{dt} \cdot t + u}$$

$$\frac{du}{dt} \cdot t + u = \frac{(ut)^2 + t(ut)}{(ut)^2 + 3t^2} = \frac{u^2t^2 + ut^2}{u^2t^2 + 3t^2}$$

$$\frac{du}{dt} \cdot t + u = \frac{u^2 + u}{u^2 + 3}$$

$$\underline{\frac{du}{dt} = \frac{u^2 + u}{u^2 + 3} \cdot \frac{1}{t} - \frac{u}{t}}$$

Good

6. Find a general solution to the differential equation $\frac{dy}{dt} = \frac{2ty}{1+t^2} + \frac{2}{1+t^2}$.

It's linear!

Rearrange: $\frac{dy}{dt} - \frac{2t}{1+t^2} \cdot y = \frac{2}{1+t^2}$

Find integrating factor: $\mu(t) = e^{\int \frac{-2t}{1+t^2} dt} = e^{-\ln|1+t^2|} = \frac{1}{1+t^2}$

Multiply: $\frac{dy}{dt} \cdot \frac{1}{1+t^2} - \frac{2t}{(1+t^2)^2} \cdot y = \frac{2}{(1+t^2)^2}$

Integrate: $\frac{1}{1+t^2} \cdot y = \int \frac{2}{(1+t^2)^2} dt + C$

Solve for y: $y = (1+t^2) \underbrace{\int \frac{2}{(1+t^2)^2} dt}_{\text{This part doesn't get any nicer}} + C(1+t^2)$

☹

7. Find the power series expansion for the general solution up to degree five to the differential equation $\frac{d^2y}{dt^2}+4y=\cos 2t$. [Hint: $\cos x$ has MacLaurin polynomial $1-x^2/2+x^4/24$.]

$x=2t$

$y=a_0+a_1t+a_2t^2+a_3t^3+a_4t^4+a_5t^5$

$y'=a_1+2a_2t+3a_3t^2+4a_4t^3+5a_5t^4$

$y''=2a_2+6a_3t+12a_4t^2+20a_5t^3$

$\cos(2t)=1-\frac{4t^2}{2}+\frac{16t^4}{24}$

$=1-2t^2+\frac{2}{3}t^4$

$2a_2+6a_3t+12a_4t^2+20a_5t^3+4a_0+4a_1t+4a_2t^2+4a_3t^3+4a_4t^4+4a_5t^5=1-2t^2+\frac{2}{3}t^4$

t^0	$2a_2+4a_0=1$	$2a_2=1-4a_0$ $\quad a_2=\frac{1}{2}-2a_0$
t^1	$6a_3+4a_1=0$	$6a_3=-4a_1$ $\quad a_3=-\frac{2}{3}a_1$
t^2	$12a_4+4a_2=-2$	$12a_4=-2-4a_2=-2-2+8a_0=-4+8a_0$ $\quad a_4=-\frac{1}{3}+\frac{2}{3}a_0$
t^3	$20a_5+4a_3=0$	$20a_5=-4a_3=\frac{8}{3}a_1$ $\quad a_5=\frac{8}{60}a_1$ $\quad a_5=\frac{2}{15}a_1$

$y(t)=a_0+a_1t+\left(\frac{1}{2}-2a_0\right)t^2-\frac{2}{3}a_1t^3+\left(\frac{-1}{3}+\frac{2}{3}a_0\right)t^4+\frac{2}{15}a_1t^5$

Excellent

8. Biff is taking Differential Equations at Enormous State University, and is having some trouble. Biff says "Crap! I do okay with the stuff where you work a problem out and get an answer, you know? But now there's this crap with, like, existing and uniqueing or something, right? So one was with $\frac{dy}{dt} = (y-2)(y-5)y$, and they said $y(0)$ is 1, and stuff is continuous and stuff, right? So then what can we say about existing and stuff?"

Explain clearly to Biff what can be said about the behavior of solutions to this differential equation based on the Existence and Uniqueness Theorems.

Well Biff, this differential equations satisfies the conditions to both theorems. By the existence theorem, we know there is exists some solution that goes through the point $y(0)=1$ and fits the differential equation.

By the uniqueness theorem, we know there is only this one solution for this initial value problem.

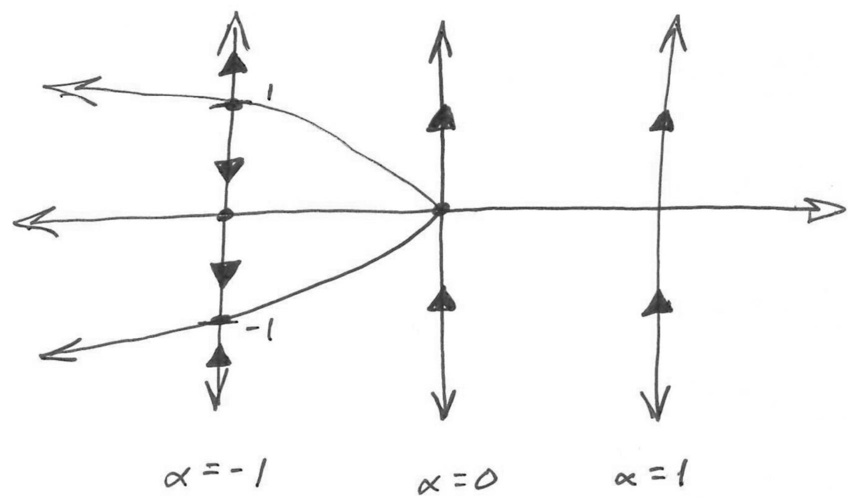
We can see that $\frac{dy}{dt}$ has equilibrium solutions of $y(t)=2$, $y(t)=5$, and $y(t)=0$. Our solution for $y(0)=1$ cannot cross any of these, because the Uniqueness theorem implies if ~~a~~ so two solutions are ever the same, ~~they~~ at a single point, then they must be the same for all of t. So, because $y(0)=1$ is between 2 and 0, our solution will forever remain trapped ~~in~~ between these two values.

Excellent!

9. **Sketch the bifurcation diagram** for the differential equation $\frac{dy}{dt} = y^4 + \alpha y^2$. Include direction arrows on the phase lines and make clear the exact α values where bifurcations occur.

$$0 = y^2(y^2 + \alpha)$$

So there's always an equilibrium when $y=0$, and for $\alpha<0$ there are also at $y = \pm\sqrt{-\alpha}$.



10. Torricelli's Law says that if water with depth $h(t)$ drains from a tank with volume $V(t)$ through a hole with area a, then the flow rate is jointly proportional to the area of the hole and the square root of the water depth. If we have a cylindrical tank with radius 2 ft and height 6 ft, and a circular hole with radius 1 in, the differential equation becomes $\frac{dh}{dt} = -\frac{1}{72}\sqrt{h}$. Solve this equation to find the height of the water after t seconds, supposing that it starts full. How long does it take the tank to drain completely?

$$\frac{1}{\sqrt{h}}\,dh = -\frac{1}{72}\,dt$$

$$\int h^{-1/2}\,dh = \int \frac{-1}{72}\,dt$$

$$2 \cdot h^{1/2} = \frac{-1}{72}t + C$$

$$\sqrt{h} = \frac{-1}{144}t + D$$

$$h = \left(D - \frac{1}{144}t\right)^2$$

If $h(0)=6$:

$$6 = \left(D - \frac{1}{144}\cdot 0\right)^2$$

$$D = \sqrt{6}$$

So $h(t) = \left(\sqrt{6} - \frac{1}{144}t\right)^2$

Then it's empty when $h(t) = 0$, or:

$$0 = \left(\sqrt{6} - \frac{1}{144}t\right)^2$$

$$0 = \sqrt{6} - \frac{1}{144}t$$

$$\frac{1}{144}t = \sqrt{6}$$

$$t = 144\sqrt{6}$$