Exam 2 Differential Equations 3/21/14

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. Find a solution to the system of differential equations

$$\frac{dx}{dt} = 3x$$

$$\frac{dy}{dt} = \qquad -2y$$

$\boxed{x = Ae^{3t} \quad y = Be^{-2t}}$

check

$\frac{d}{dt}(Ae^{3t}) = 3Ae^{3t} \rightarrow 3Ae^{3t} \overset{?}{=} 3Ae^{3t}$ ✓

Great

$\frac{d}{dt}(Be^{-2t}) = -2Be^{-2t} \rightarrow -2Be^{-2t} \overset{?}{=} -2Be^{-2t}$ ✓

2. Determine whether $x(t) = 4e^{4t}$, $y(t) = e^{4t}$ is a solution to the system

$$\frac{dx}{dt} = 3x + 4y$$

$$\frac{dy}{dt} = \ x + 0y$$

$\frac{d(4e^{4t})}{dt} \overset{?}{=} 3(4e^{4t}) + 4(e^{4t})$

$16e^{4t} \overset{?}{=} 12e^{4t} + 4e^{4t}$

$16e^{4t} = 16e^{4t}$ ✓

$\frac{d(e^{4t})}{dt} \overset{?}{=} 4e^{4t}$

$4e^{4t} = 4e^{4t}$ ✓

Since $(4e^{4t}, e^{4t})$ is a solution for both $\frac{dx}{dt}$ and $\frac{dy}{dt}$, it is a solution for the system.

Good

3. Construct a system of differential equations, with all coefficients representing positive constants, to model the interaction of two populations where:
   - The first population would experience exponential growth in the absence of the second
   - Interaction between the two populations hurts the first population
   - The second population would experience logistic growth in the absence of the first
   - Interaction between the two populations benefits the second population

$$\frac{dx}{dt} = +\alpha x - \beta xy$$

$$\frac{dy}{dt} = \gamma y\left(1-\frac{y}{K}\right) + \delta xy$$

K = carrying capacity

α, β, γ, and δ are all proportionality constants.

Excellent!

4. Find all equilibria of the system of differential equations:

$$\frac{dr}{dt} = 0.01r(25-r) - 0.04rm$$

W

$$\frac{dm}{dt} = -0.2m + 0.02rm$$

$0 = 0.01r(25-r) - 0.04rm = 0.01r(25-r-4m)$

$0 = -0.2m + 0.02rm = 0.02m(-10+r)$

Either ① $0.02m = 0$ or ② $-10+r = 0$

① So if $0.02m = 0$

Then $m = 0$ and either:

$0.01r = 0$ so $r = 0$

or

$25 - r - 4(0) = 0$

so $25 = r$

② If $(-10+r) = 0$

Then $R = 10$ and

$25 - 10 - 4m = 0$

$15 = 4m$

$m = 15/4$

Great

(r, m) equilibria

$\boxed{(0,0),\ (25,0),\ (10, \tfrac{15}{4})}$

5. Consider the system

$$\frac{dS}{dt} = -0.1SI$$

$$\frac{dI}{dt} = 0.1SI - 0.5I \, .$$

$$\frac{dR}{dt} = 0.25I$$

Use Euler's method with a step size of $\Delta t = 0.1$ to project $S(0.1)$, $I(0.1)$, and $R(0.1)$ if $S(0) = 60$, $I(0) = 4$, and $R(0) = 0$.

t	S	$\frac{dS}{dt}$	ΔS	I	$\frac{dI}{dt}$	ΔI	R	$\frac{dR}{dt}$	ΔR
0	60	-24	-2.4	4	22	2.2	0	1	0.1
.1	57.6			6.2			0.1		

At $t=0$, $\frac{dS}{dt} = -0.1(60)(4) = -24$ $\quad -24(.1) = -2.4$

$\frac{dI}{dt} = 0.1(60)(4) - 0.5(4) = 22$ $\quad 22(.1) = 2.2$

$\frac{dR}{dt} = 0.25(4) = 1$ $\quad 1(.1) = .1$

Nice

$S(0.1) = 57.6$

$I(0.1) = 6.2$

$R(0.1) = 0.1$

6. Consider the equation

$$\frac{d^2y}{dt^2} + \frac{k}{m}y = 0$$

for the motion of a simple harmonic oscillator. Consider the function $y(t) = \cos \beta t$. Under what conditions on β is $y(t)$ a solution?

$y'(t) = -\beta \sin \beta t$

$y''(t) = -\beta^2 \cos \beta t$

$-\beta^2 \cos \beta t + \frac{k}{m} \cos \beta t = 0$

$\left(-\beta^2 + \frac{k}{m}\right) \cos \beta t = 0$

$\cos \beta t$ is only sometimes 0, so $-\beta^2 + \frac{k}{m}$ must always equal 0.

$-\beta^2 + \frac{k}{m} = 0$

$\frac{k}{m} = \beta^2$

$\boxed{\pm\sqrt{\frac{k}{m}} = \beta}$

Great

7. Suppose $y(t)$ is a solution to the differential equation

$$\frac{d^2y}{dt^2}+\alpha\frac{dy}{dt}+\beta y=0.$$

What can you say about $k \cdot y(t)$, where k is a constant?

properties of derivatives says $\frac{d(ky)}{dt} = \frac{dy}{dt}(k)$

$$\frac{d^2(ky)}{dt^2} = \frac{d^2y}{dt}(k)$$

$\frac{d^2y}{dt^2} + \alpha \frac{dy}{dt} + B(y) = 0$ is true since $y(t)$ is a solution

$$\frac{d^2(ky)}{dt^2} + \alpha \frac{d(ky)}{dt} + B(ky) \stackrel{?}{=} 0$$

by prop. of der.,

$$k\left(\frac{d^2y}{dt^2}\right) + k(\alpha)\left(\frac{dy}{dt}\right) + k(B)(y) \stackrel{?}{=} 0$$

$k\left(\frac{d^2y}{dt^2} + \alpha\frac{dy}{dt} + By\right) \stackrel{?}{=} 0 \Rightarrow \underline{k(0)=0}$ k Solution!

Since is equal to $\varnothing$, then $k \cdot \varnothing$ is still $\varnothing$

and $\underline{k \cdot (y(t))}$ is a solution. Nice!

8. Find a non-trivial solution to the system

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = 5x.$$

$$\frac{d^2x}{dt^2} = 2\frac{dx}{dt} + 3\frac{dy}{dt}$$

$$x'' = 2x' + 3(5x)$$

$$x'' = 2x' + 15x$$

probably $x = e^{st}$, $x' = se^{st}$, $x'' = s^2e^{st}$.

So

$$s^2e^{st} = 2se^{st} + 15e^{st}$$

$$y' = 5(k_1e^{5t} + k_2e^{-3t})$$

$$y = k_1e^{5t} - \frac{5}{3}k_2e^{-3t}$$

$$0 = s^2e^{st} - 2se^{st} - 15e^{st}$$

$$0 = e^{st}(s^2 - 2s - 15)$$

Since $e^{st} \neq 0$ then

$$\boxed{\begin{aligned} x &= k_1e^{5t} + k_2e^{-3t} \\ y &= k_1e^{5t} - \frac{5}{3}k_2e^{-3t} \end{aligned}}$$

$$0 = s^2 - 2s - 15$$

$$0 = (s-5)(s+3)$$

$s = 5$ or $s = -3$

Excellent!

9. Find a general solution to the system

$$\frac{dx}{dt} = 2x + 3y$$

$$\frac{dy}{dt} = -4y.$$

Since $\frac{dy}{dt} = -4y$ $\boxed{y = Ae^{-4t}}$ Plug it into $\frac{dx}{dt}$

$\frac{dx}{dt} = 2x + 3(Ae^{-4t})$

$\frac{dx}{dt} - 2x = 3Ae^{-4t}$ this is linear

integrating factor

$u = e^{\int -2dt} = e^{-2t}$

Multiply each side by u

$e^{-2t}\left(\frac{dx}{dt} - 2x\right) = e^{-2t} \cdot 3Ae^{-4t}$

and integrate

$e^{-2t}x = \int e^{-2t} \cdot 3Ae^{-4t}dt$

$e^{-2t}x = \int 3Ae^{-6t}dt$

$e^{-2t}x = 3A\int e^{-6t}dt$

$e^{-2t}x = 3A \cdot \left(\frac{-1}{6}\right)e^{-6t} + c$

$(e^{2t})e^{-2t}x = \left(-\frac{1}{2}Ae^{-6t} + c\right)(e^{2t})$

$x = -\frac{1}{2}Ae^{-4t} + ce^{2t}$ Well done

$y = Ae^{-4t}$

10. Find a general solution to the system

$$\frac{dx}{dt} = 2x + 3y$$
$$\frac{dy}{dt} = \alpha y.$$

Also partially decoupled

$$\int \frac{dy}{y} = \int \alpha \, dt$$
$$\ln|y| = \alpha t + c$$
$$y = k_1 e^{\alpha t}$$

Linear

$$\frac{dx}{dt} - 2x = 3k_1 e^{\alpha t}$$

$$\mu(t) = e^{\int -2dt} = e^{-2t}$$

Yes!

$$\frac{dx}{dt}\left(e^{-2t}\right) - 2x\left(e^{-2t}\right) = 3k_1 e^{(\alpha-2)t}$$
$$\frac{d}{dt}\left(x \cdot e^{-2t}\right) = 3k_1 e^{(\alpha-2)t}$$
$$x \cdot e^{-2t} = \frac{3k_1}{(\alpha-2)} e^{(\alpha-2)t} + k_2$$
$$x = \frac{3k_1}{(\alpha-2)} e^{(\alpha)t} + k_2 e^{2t}$$

Solution

$$x(t) = \frac{3k_1}{(\alpha-2)} e^{\alpha t} + k_2 e^{2t}$$
$$y(t) = k_1 e^{\alpha t}$$

To check with Problem #9, if $\alpha = -4$,

$$x(t) = -\frac{1}{2} k_1 e^{-4t} + k_2 e^{2t}$$
$$y(t) = k_1 e^{-4t}$$

Matches perfectly with result from #9!