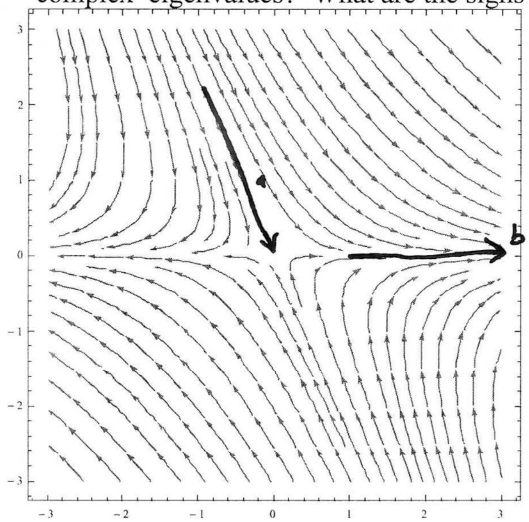
Exam 3 Differential Equations 4/8/16

Each problem is worth 10 points. For full credit indicate clearly how you reached your answer.

1. Does the planar system whose phase plane is shown below have two real eigenvalues, or two complex eigenvalues? What are the signs on the real portions of the eigenvalues?



Two real eigen values; one positive, and one negative because it is a saddle.

Good

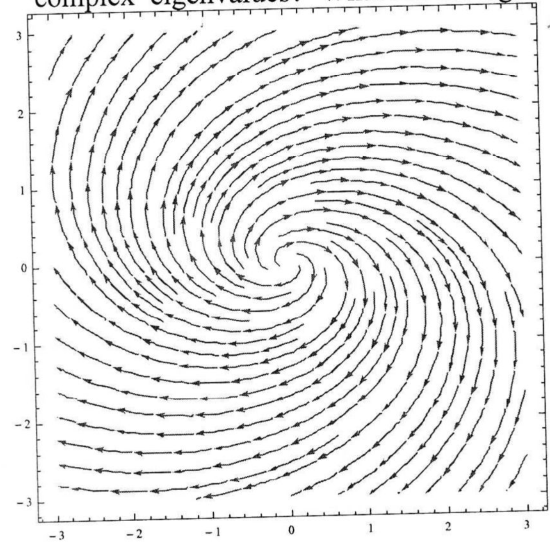
2. Estimate two (non-parallel) eigenvectors of the planar system whose phase plane is shown above.

drawn in above are two nonparallel eigen vector approximations:

$\vec{V}_a = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\vec{V}_b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Excellent!

3. Does the planar system whose phase plane is shown below have two real eigenvalues, or two complex eigenvalues? What are the signs on the real portions of the eigenvalues?



Two complex eigenvalues

Real portion is positive b/c it moves away from the origin

Great!

4. Suppose we have a planar system for which $\mathbf{Y}(t) = 2e^{3t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + 5e^{-t}\begin{pmatrix} -1 \\ 0 \end{pmatrix}$ is a solution.

Find a solution passing through the point (1,0).

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$\hat{y} = Ae^{3t}\begin{pmatrix} 2 \\ -1 \end{pmatrix} + Be^{-t}\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

Easiest way;

pass through (1,0) at time t=0.

If $Y_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} A(2) + B(-1) \\ A(-1) + B(0) \end{pmatrix}$

$1 = 2A - B \qquad 1 = -B, \; B = -1$

$0 = -A, \; 0 = A$

Good

$\hat{y} = -e^{-t}\begin{pmatrix} -1 \\ 0 \end{pmatrix}$

5. Is $y = t\,e^{\lambda t}$ a solution to the differential equation $y'' - 2\lambda\, y' + \lambda^2 y = 0$?

Let's check.

$$y = t e^{\lambda t}$$

$$y' = \lambda t e^{\lambda t} + e^{\lambda t}$$

$$y'' = \lambda^2 t e^{\lambda t} + \lambda e^{\lambda t} + \lambda e^{\lambda t} = \lambda^2 t e^{\lambda t} + 2\lambda e^{\lambda t}$$

Plug it in:

$$\lambda^2 t e^{\lambda t} + 2\lambda e^{\lambda t} - 2\lambda(\lambda t e^{\lambda t} + e^{\lambda t}) + \lambda^2 (t e^{\lambda t}) \stackrel{?}{=} 0$$

$$\lambda^2 t e^{\lambda t} + 2\lambda e^{\lambda t} - 2\lambda^2 t e^{\lambda t} - 2\lambda e^{\lambda t} + \lambda^2 t e^{\lambda t} \stackrel{?}{=} 0$$

$$t e^{\lambda t}(\lambda^2 - 2\lambda^2 + \lambda^2) + e^{\lambda t}(2\lambda - 2\lambda) \stackrel{?}{=} 0$$

$$t e^{\lambda t}(0) + e^{\lambda t}(0) = 0.$$

Because we plugged it in and it works,

$y = t e^{\lambda t}$ is a solution.

Excellent!

6. Consider the system $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & 6 \\ 1 & -2 \end{pmatrix}\mathbf{Y}$. Find a general solution to this system.

$\det(A-\lambda I) = \det\begin{pmatrix} 3-\lambda & 6 \\ 1 & -2-\lambda \end{pmatrix} = 0$

$(3-\lambda)(-2-\lambda) - 6(1) = 0$

$\lambda^2 - \lambda - 6 - 6 = 0$

$(\lambda+3)(\lambda-4) = 0 \Rightarrow \lambda = -3, 4$

If $\lambda = -3$

$6x + 6y = 0$
$x + y = 0$
$\rightarrow x = -y$

$\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

If $\lambda = 4$

$-x + 6y = 0$
$x - 6y = 0$
$\rightarrow x = 6y$

$\begin{bmatrix} 6 \\ 1 \end{bmatrix}$

$\hat{Y} = Ae^{-3t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + Be^{4t}\begin{pmatrix} 6 \\ 1 \end{pmatrix}$

Good Job.

Thanks to the Bandicoot Theorem and the Linearity Principle.

7. State and prove the Bandicoot Theorem.

Bandicoots are marsupials!

For a differential equation of the form $\frac{d\hat{Y}}{dt} = \hat{A}\hat{Y}$

for a matrix $\hat{A}$ with eigenvalue λ and corresponding

eigenvector $\bar{v}$, $\hat{Y} = e^{\lambda t}\bar{v}$ is a solution.

Proof: $\hat{Y} = e^{\lambda t}\bar{v}$

$$\frac{d\hat{Y}}{dt} = \lambda e^{\lambda t}\bar{v}$$

$$= \lambda \bar{v} e^{\lambda t}$$

$$= \hat{A}\bar{v} e^{\lambda t}$$

(By the definitions of eigenvalues and eigenvectors)

$$= \hat{A} e^{\lambda t}\bar{v}$$

$$= \hat{A}\hat{Y}$$

Because we plugged it in and it works,
$\hat{Y} = e^{\lambda t}\bar{v}$ is a solution.

Great

8. Consider the system $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} 6 & -5 \\ 5 & -4 \end{pmatrix}\mathbf{Y}$. Find a solution to this system satisfying the initial condition $\mathbf{Y}(0) = (0,1)$.

$$\begin{pmatrix} 6-\lambda & -5 \\ 5 & -4-\lambda \end{pmatrix} = 0 \Rightarrow (6-\lambda)(-4-\lambda) - \overbrace{(-5)(5)}^{-25} = 0$$

$$-24 + 4\lambda - 6\lambda + \lambda^2 + 25 \Rightarrow \lambda^2 - 2\lambda + 1$$

$$(\lambda-1)(\lambda-1) = 0 \quad \text{so} \quad \underline{\lambda_1 = 1 \ \& \ \lambda_2 = 1} \quad \underline{\text{Repeated Eigenvalues}}$$

Use <u>Great Theorem of Pg 305</u>

$$\hat{Y}(t) = e^{\lambda t} \cdot V_0 + te^{\lambda t} \cdot V_1 \quad \text{where} \quad V_0 = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

$$\underline{V_1 = (A - \lambda I)V_0}$$

$$V_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$V_1 = \begin{pmatrix} 6 & -5 \\ 5 & -4 \end{pmatrix} - \overbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}^{\lambda I} V_0 \Rightarrow \begin{pmatrix} 5 & -5 \\ 5 & -5 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & -5 \\ 0 & -5 \end{pmatrix} = \begin{pmatrix} -5 \\ -5 \end{pmatrix}$$

$$\text{So} \quad \underline{\hat{Y}(t) = e^t \begin{pmatrix} 0 \\ 1 \end{pmatrix} + te^t \begin{pmatrix} -5 \\ -5 \end{pmatrix}}$$

Excellent!

9. Consider the system $\dfrac{d\mathbf{Y}}{dt} = \begin{pmatrix} 3 & b \\ 1 & -3 \end{pmatrix}\mathbf{Y}$. For what value(s) of b will solutions form closed curves, i.e. return to the initial condition after some amount of time?

$$\frac{dY}{dt} = \begin{pmatrix} 3 & b \\ 1 & -3 \end{pmatrix} Y$$

$$\det\begin{pmatrix} 3-\lambda & b \\ 1 & -3-\lambda \end{pmatrix} \Rightarrow (3-\lambda)(-3-\lambda) - b = 0$$

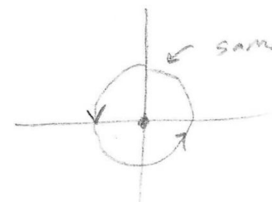
$$\lambda^2 - 9 - b = 0$$

$$\lambda^2 + (-b-9) = 0$$

return to the initial condition meaning a center? I believe

$\lambda = \pm bi$ so all imaginary

same initial conditio



For $\lambda^2 + (-b-9) = 0$ to yield only imaginary eigen values $(-b-9)$ must be greater than 0.

$$-b - 9 > 0$$

$$-b > 9 \quad \text{Yes.}$$

$$b < -9$$

So all b values must be less than -9 for our eigenvalues to be strictly imaginary and to yield a center where solutions form closed curves!

Check: $\lambda = -10$

$$\frac{dY}{dt} = \begin{pmatrix} 3 & -10 \\ 1 & -3 \end{pmatrix} Y \quad \det\begin{pmatrix} 3-\lambda & -10 \\ 1 & -3-\lambda \end{pmatrix}$$

$$(-3-\lambda)(3-\lambda) + 10 = 0$$

$$\lambda^2 - 9 + 10 = 0$$

$$\lambda^2 = -1$$

$$\lambda = \pm i \quad \text{works!}$$

Nice touch!

10. Find the general solution to the linear system $\frac{d\mathbf{Y}}{dt} = \begin{pmatrix} -2 & 9 \\ -2 & 4 \end{pmatrix}\mathbf{Y}$.

$$\begin{bmatrix} -2-\lambda & 9 \\ -2 & 4-\lambda \end{bmatrix} \vec{Y} = \vec{0}$$

$$(-2-\lambda)(4-\lambda) + 18 = 0$$

$$\lambda^2 - 2\lambda + 10 = 0$$

$$\lambda = \frac{2 \pm \sqrt{4-40}}{2} = \underline{1 \pm 3i}$$

If $\lambda = 1+3i$

$$-2x + 9y = (1+3i)x \qquad 9y = (3+3i)x$$

$$-2x + 4y = (1+3i)y \qquad 3y = (1+i)x$$

$$\underline{\begin{bmatrix} 3 \\ 1+i \end{bmatrix}}$$

$$\vec{Y}(t) = e^{(1+3i)t}\begin{bmatrix} 3 \\ 1+i \end{bmatrix} = e^{t}e^{3it}\begin{bmatrix} 3 \\ 1+i \end{bmatrix} = e^{t}\left[\cos(3t) + i\sin(3t)\right]\begin{bmatrix} 3 \\ 1+i \end{bmatrix}$$

$$= e^{t}\begin{bmatrix} 3\cos(3t) + i\,3\sin(3t) \\ \cos(3t) + i\sin(3t) + i\cos(3t) - \sin(3t) \end{bmatrix} = e^{t}\begin{bmatrix} 3\cos(3t) \\ \cos(3t) - \sin(3t) \end{bmatrix} + ie^{t}\begin{bmatrix} 3\sin(3t) \\ \sin(3t) + \cos(3t) \end{bmatrix}$$

By the Complex Miracle Theorem:

$$\underline{\vec{Y}(t) = Ae^{t}\begin{bmatrix} 3\cos(3t) \\ \cos(3t) - \sin(3t) \end{bmatrix} + Be^{t}\begin{bmatrix} 3\sin(3t) \\ \sin(3t) + \cos(3t) \end{bmatrix}}$$

is a general solution.

well done