Examlet 1 Foundations of Advanced Math 2/4/16

1. Show that the sum of two odd integers is even.

So, $m \in \mathbb{Z}$ is odd which means $m = 2x+1$ where $x \in \mathbb{Z}$ and $n \in \mathbb{Z}$ is odd which means $n = 2y+1$ where $y \in \mathbb{Z}$

$m+n = 2x+1+2y+1 = 2x+2y+2 = 2(x+y+1)$ we know that $x+y+1$ is an integer by closure, so $m+n$ is even by definition. $\square$

Excellent.

2. Determine whether $P \vee Q$ is logically equivalent to $\neg(\neg P \wedge \neg Q)$.

P	Q	$P \vee Q$	$\neg P$	$\neg Q$	$\neg P \wedge \neg Q$	$\neg(\neg P \wedge \neg Q)$
T	T	T	F	F	F	T
T	F	T	F	T	F	T
F	T	T	T	F	F	T
F	F	F	T	T	T	F

Since $P \vee Q$ and $\neg(\neg P \wedge \neg Q)$ have the same truth values under all circumstances, they are logically equivalent. $\square$

Excellent!

3. If $a \equiv_n 1$, and $b \equiv_n 1$ then $a \equiv_n b$.

Well, take a to be congruent modulo n to 1 and b to be congruent modulo n to 1. From here, we can say that $n \mid 1-a$ and $n \mid 1-b$. Futhermore, we can say that $1-a = n \cdot k$ and $1-b = n \cdot q$ where $k, q \in \mathbb{Z}$.

$a = 1-nk$ $b = 1-nq$ $b-a = 1-nq-(1-nk)$ $b-a = 1-nq-1+nk$ $b-a = -nq+nk$ $b-a = n(-q+k)$. Because q and k are integers, $-q+k$ is an integer by closure. Therefore, b-a is n times an integer so we can say that $n \mid b-a$. This means that we can also say that $a \equiv_n b$. $\square$

Good

4. $\sqrt{3}$ is irrational.

Suppose $\sqrt{3}$ were rational such that $\sqrt{3} = p/q$ for some integers p and q, and that p and q were reduced to have no common factors. So,

$\sqrt{3} = p/q$, and squaring both sides, we get

$3 = p^2/q^2$, now multiply both sides by q^2 so

$3q^2 = p^2$. (q^2) is an integer by closure of integers under multiplication, so $3q^2$ is threven. This means p^2 is threven, and from previous exercises,* we know p must also be threven. So, $p = 3r$, where $r \in \mathbb{Z}$. Substituting $p = 3r$ into $3q^2 = p^2$, we get $3q^2 = (3r)^2$. This simplifies to $3q^2 = 9r^2$. Dividing both sides by 3, we get $q^2 = 3r^2$. (r^2) is an integer by closure of integers under multiplication, so $3r^2$ is threven by definition. That means q^2 is threven, and from previous exercises,* we know q is also threven. Since p and q are both threven, they would share a common factor of 3, contradicting our supposition of no common factors, and leading us to conclude $\sqrt{3}$ is irrational. □

* previous exercises showed a number is threven, throdd, or throddodd. These showed the square of a threven was threven, while the square of a throdd integer was throdd, and the square of a throddodd integer was also throdd. Since these are the only three cases, if a number squared is threven, the number itself must also be threven

Nice!

5. Recall that if C is a set of real numbers, we say b is an **upper bound** for C iff $\forall x \in C, b \geq x$. Show that any collection of exactly n distinct real numbers (where n is a natural number) has an upper bound.

We'll proceed by induction to prove that any collection of n distinct real numbers has an upper bound. We'll start with base case $n=1$, a set with 1 real number. That number, z, is the only number, meaning it is the largest number in the set, so $b=z$, and the statement is true for $n=1$.

Now assume the statement is true for $n=k$, such that there are k real numbers in the set, and $\forall x \in C$, $b \geq x$. We now must prove the statement is true for $k+1$.

We add one real number, y, to set k, so there are now $k+1$ natural numbers in the set. Here, y could be one of three cases, $y > b$, $y < b$, or $y = b$. If $y > b$, $\forall x \in C$, $y > b > x$, so $y > x$; y becomes the new upper bound. Next, $y < b$, $y \in C$, $\forall x \in C$, $b \geq x$, so b remains the upper bound. Next, if $y = b$, $y \in C$, $\forall x \in C$, $b \geq x$, the statement remains true, so there is still an upper bound. In all three cases, the statement is true, there is an upper bound, meaning for $n=k+1$, n real numbers has an upper bound.

Therefore, since the statement is true for base case $n=1$, and when the statement is true for $n=k$, it is also true for $n=k+1$, we can say by mathematical induction, that a set of n distinct real numbers has an upper bound. □

Nice!