Examlet 4 Foundations of Advanced Math 4/15/16

1. a) State the definition of a relation from A to B.

A relation from A to B can be defined as a subset of $A \times B$.

Good

b) State the definition of the degree of a vertex v in a graph.

The degree of a vertex v in a graph can be defined as the number of edges the vertex v has.

Good

c) State the definition of a tree.

A tree can be defined as a connected graph without any cycles.

Great

2. Consider the relation ~ on $\mathbb{Z}$ defined by $a \sim b$ iff $5|(a-b)$. Show that ~ is an equivalence relation, being clear about your reasoning.

$\sim$ is an equivalence relation iff it is reflexive, symmetric, and transitive.

Reflexive: Take $a \in \mathbb{Z}$, so $5|(a-a)$, or $(a-a) = 5n$, $n \in \mathbb{Z}$. Since $(a-a) = 0 = 5(0)$, and $0 \in \mathbb{Z}$, so $a \sim a$, so $\sim$ is reflexive.

Symmetric: Suppose $a \sim b$, so $5|(a-b)$, or $(a-b) = 5n$, $n \in \mathbb{Z}$. Since $n \in \mathbb{Z}$, $-n \in \mathbb{Z}$ and $(b-a) = 5(-n)$, so $b \sim a$, so $\sim$ is symmetric.

Transitive: Suppose $a \sim b$ and $b \sim c$, so $5|(a-b)$ and $5|(b-c)$, so $(a-b) = 5n$, $n \in \mathbb{Z}$ and $(b-c) = 5m$, $m \in \mathbb{Z}$. adding these equations gives $(a-b)+(b-c) = (5n)+(5m)$, or $(a-c) = 5(n+m)$, $n+m \in \mathbb{Z}$ by C.o.I, so $a \sim c$, so $\sim$ is transitive.

Since we have shown that $\sim$ is reflexive, symmetric, and transitive, we have also shown that $\sim$ is an equivalence relation. $\square$

Nice

3. a) Express the definition of the sum of two functions function formally in terms of ordered pairs.

Let f and g be two functions.

$f+g = \{(a, b+c) \mid (a,b) \in f, (a,c) \in g\}$

Great

b) Express the definition of the composition of two functions formally in terms of ordered pairs.

Let f and g be two functions.

$f \circ g = \{(a,c) \mid (a,b) \in g, (b,c) \in f\}$

Excellent!

4. a) Let S be a set and Π a partition of S defined by $a \sim b \Leftrightarrow \exists P \in \Pi$ for which $a, b \in P$.
Then $\sim$ is a reflexive relation.

Let $a \in S$, π is a partition of S so
the union of all the sets in π is all of S, so
$\exists P \in \pi$ where $a \in P$, so $a \sim a$ therefore it
is a reflextve relation. □

Great

b) Let S be a set and Π a partition of S defined by $a \sim b \Leftrightarrow \exists P \in \Pi$ for which $a, b \in P$.
Then $\sim$ is a symmetric relation.

If $a \sim b$, then $a, b \in P$ where $P \in \Pi$. Since
P is a set, because a partition is a set of nonempty
pairwise disjoint sets, then $b, a \in P$ so $b \sim a$
therefore it is a symmetric relation. □

Good

c) Let S be a set and Π a partition of S defined by $a \sim b \Leftrightarrow \exists P \in \Pi$ for which $a, b \in P$.
Then $\sim$ is a transitive relation.

If $a \sim b$, then $a, b \in P_1$ where $P_1 \in \Pi$ and if
$b \sim c$, then $b, c \in P_2$ where $P_2 \in \pi$. P_1 and
P_2 are pairwise disjoint by definition of a
partition, so either $P_1 \cap P_2 = \emptyset$ or $P_1 = P_2$.
Since $b \in P_1 \cap P_2$, $P_1 = P_2$ so $a, b, c \in P_1$
so $a \sim c$, therefore it is a transitive relation. □

Nice!

5. Let G be a graph and say two vertices u and v of G are related iff u and v are joined by a walk of odd length. Is this relation reflexive? symmetric? transitive? Support your answers well. number of edges

Reflexive

Take graph G, and $v \in G$. Take $G = \bullet v$ $v \nsim v$ because v and v are not joined by a walk of odd length, so $\sim$ is not reflexive.

Great

Symmetric

Take graph G and $u, v \in G$ and $u \sim v$, so u and v are joined by a walk of odd length, meaning there is a sequence alternating vertices and edges, where each edge is adjacent to the preceeding and succeeding vertices, starting with u and ending with v. Doing this walk in reverse, starting at v and ending with u, would also have an odd length, so there is a walk of odd length joining v and u, so $v \sim u$, and $\sim$ is symmetric. Nice!

Transitive

Take graph G and $u, v, w \in G$.

Let $G =$ 

- $u \sim v$ because u and v are joined by a walk of odd length
- $v \sim w$ because v and w are joined by a walk of odd length
- $u \nsim w$ because u and w are not joined by a walk of odd length,

so since $u \sim v$ and $v \sim w$, but $u \nsim w$, $\sim$ is not a transitive relation.

Well done.

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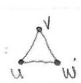
+2

Note There are graphs where $\sim$ is reflexive and transitive. Let $G =$  In this case, $\sim$ is reflexive and transitive, but in general, this relation is not always reflexive and transitive.