Examlet 3a Foundations of Advanced Math 3/24/17

1. a) State the definition of sets A and B being **equipollent**.

A and B are equipollent if and only if there <u>exists</u> a <u>bijection</u> from A to B.

b) Give five distinct examples of sets equipollent to $\mathbb{N}$.

$\underline{\mathbb{Z}}$, $\underline{\mathbb{Q}}$, the set of <u>odd naturals</u>, the set of <u>even naturals</u>, $\underline{\mathbb{Z} \times \mathbb{Z}}$

<u>Great</u>

2. The composition of two injective functions is injective.

Let $f: A \to B$ be injective so
$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

Let $g: B \to C$ be injective so
$g(a_1) = g(a_2) \Rightarrow a_1 = a_2$

Suppose $g \circ f(a_1) = g \circ f(a_2)$

Rewritten: $g(f(a_1)) = g(f(a_2))$

Because g is injective, $g(f(a_1)) = g(f(a_2)) \Rightarrow f(a_1) = f(a_2)$

Because f is injective, $f(a_1) = f(a_2) \Rightarrow a_1 = a_2$

So $g(f(a_1)) = g(f(a_2)) \Rightarrow a_1 = a_2$

So by definition, the composition of two injective functions is injective. $\square$

Excellent!

3. The composition of two surjective functions is surjective.

Let $f: A \to B$ be surjective, so
$\forall\, b \in B$, $\exists\, a \in A$ such that $f(a) = b$

Let $g: B \to C$ be surjective, so
$\forall\, c \in C$, $\exists\, b \in B$ such that $g(b) = c$

So we know $\forall c \in C$, $\exists b \in B$ such that $g(b) = c$
Taking that some b,
we know for that $b \in B$, $\exists a \in A$ such that $f(a) = b$

So $g(f(a)) = c$ $\quad g(b) = c$

So $\forall c \in C$, $\exists a \in A$ such that $g(f(a)) = c$

So by definition, the composition of two surjective functions is surjective. $\square$

Excellent!

4. Let $f(x)=\sqrt{4+2x}$. What is the inverse function for *f*, and what are its domain and codomain?

$y=\sqrt{4+2x}$

$y^2=4+2x$

$2x=y^2-4$

$x=\frac{y^2-4}{2}$

We have $f(x)=\sqrt{4+2x}$, and we are not allowed to take the square root of negative numbers

Therefore, the domain of f are all real numbers from -2 to infinity : $D=[-2;+\infty)$

Codomain of f are all positive reals and 0 : $[0;+\infty)$

Therefore, the inverse function is $g(x)=\frac{x^2-4}{2}$

The domain of g are all positive reals and 0 : $[0;+\infty)$

The codomain of g are all real number from -2 to infinity : $[-2;+\infty)$

Excellent!

5. a) Let $f_1: \mathbb{R} \to \mathbb{R}$ and $f_2: \mathbb{R} \to \mathbb{R}$ be even functions. Then $f_1 + f_2$ is an even function.

If $f_1: \mathbb{R} \to \mathbb{R}$ is an even function, then $\forall x \in \mathbb{R}$, $f_1(-x) = f_1(x)$

If $f_2: \mathbb{R} \to \mathbb{R}$ is an even function, then $\forall x \in \mathbb{R}$, $f_2(-x) = f(x)$

By definition, $(f_1 + f_2)(x) = f_1(x) + f_2(x)$

So $(f_1+f_2)(-x) = \underbrace{f_1(-x) + f_2(-x) = f_1(x) + f_2(x)}_{\text{because } f_1 \text{ and } f_2 \text{ are even function}} = (f_1 + f_2)(x)$

We have: $\forall x \in \mathbb{R}$, $(f_1+f_2)(-x) = (f_1+f_2)(x)$

Therefore, $f_1 + f_2$ is even by definition

Good

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b) Let $n \in \mathbb{N}$, and $f_i: \mathbb{R} \to \mathbb{R}$ be an even function for each $i \in \{x \in \mathbb{N} \mid 1 \le x \le n\}$. Then $\sum_{i=1}^{n} f_i$ is an even function.

Proceed by induction.

Base case, $n = 1$: $\sum_{i=1}^{1} f_i = f_1$, and we know that $f_1: \mathbb{R} \to \mathbb{R}$ is even, so $\sum_{i=1}^{1} f_i$ is even

Now suppose that the statement is true for $k \in \mathbb{N}$, therefore $\sum_{i=1}^{k} f_i$ is an even function (we just suppose it)

Now let's look at $k+1$:

$\sum_{i=1}^{k+1} f(i) = \sum_{i=1}^{k} f(i) + f(k+1)$, so $\sum_{i=1}^{k} f(i)$ is even according to our assumption

$f(k+1)$ is even by our initial condition $((k+1) \in \mathbb{N})$

Both $\sum_{i=1}^{k} f(i)$ and $f(k+1)$ are even, so their sum is even as well (according to ex. 5 (a))

Therefore, if the statement is true for k, then it is also true for $k+1$

So by induction $\sum_{i=1}^{n} f_i$ is an even function for $i \in \{x \in \mathbb{N}, 1 \le x \le n\}$, $n \in \mathbb{N}$

Great