1. If $a \equiv_3 2$, then $a^2 \equiv_3 1$.

$a \equiv_3 2 \rightarrow 3 \mid 2-a \rightarrow 3n = 2-a$ where $n \in \mathbb{Z}$

$a = 2-3n$

$a^2 = (2-3n)^2 = (2-3n)(2-3n)$

$a^2 = 4 - 6n - 6n + 9n^2$

$a^2 = 9n^2 - 12n + 3 + 1$

$a^2 = 3(3n^2 - 4n + 1) + 1$

$a^2 - 1 = 3\underbrace{(3n^2 - 4n + 1)}$

↳ an integer by closure $\therefore$

$3 \mid a^2 - 1 \therefore a^2 \equiv_3 1$

So if $a \equiv_3 2$, then $a^2 \equiv_3 1$. $\square$

Great

Scrap work

Want to prove:

$a^2 \equiv_3 1 \rightarrow 3 \mid 1 - a^2 \rightarrow$

$3z = 1 - a^2$ where $z \in \mathbb{Z}$

1 2. Consider each of the following statements. Tell whether each is true or false, and justify your conclusion.

(a) $(\exists x \in \mathbb{R})(\exists y \in \mathbb{R}), x + y = 0.$

This is true because the statement is saying that there is a real number x that exists and a real number y that

3 exits where adding them will equal zero

example: $2 + (-2) = 0$ Good

(b) $(\forall x \in \mathbb{R})(\exists y \in \mathbb{R}), x + y = 0.$

3 This is true because for all real numbers there exists that number's negative such that adding them will equal zero.

example: $-4.2 + 4.2 = 0$ Great

(c) $(\forall x \in \mathbb{R})(\forall y \in \mathbb{R}), x + y = 0.$

3 This is false because for all real numbers, not all real numbers give zero when added together.

example: $2 + 4 \neq 0$

Great

3. For any $n \in \mathbb{N}, 3 \mid n^3 - n$

1 3 3 1

Proof: Proceeding with induction

Base Case $n=0$ $\quad 0^3-0=0 \quad 0=3(0)$ ✓

$n=1 \quad 1^3-1=0$ ✓

$n=2 \quad 2^3-2=6 \quad 6=3(2)$ ✓

Assume the proposition is true for $n=k$

$k^3-k=3x$

Scratch work:

$n=k+1$

$(k+1)^3-(k+1)$

$k^3+3k^2+3k+\cancel{1}-k-\cancel{1}$

k^3-k+3k^2+3k

$+\underbrace{3(k^2+k)}_{\text{int by closure called } y}$

$y \in \mathbb{Z}$

When we use $k+1=n$ in the formula we get:

$\underbrace{k^3-k}+3k^2+3k-\cancel{1}+\cancel{1}$

This porition is equal to our inductive hypothesis so lets put that in. We now get

$3x+3k^2+3k$ If we factor out 3:

$3\underbrace{(x+k^2+k)}_{\text{integer by closure called } y}$

So for $n=k+1$ $\quad (k+1)^3-(k+1)=3y$

meaning that $3 \mid (k+1)^3-(k+1)$

Using base cases and our inductive hypothesis, we have proved by induction that $3 \mid n^3-n$ for any $n \in \mathbb{N}$

Great

4. Determine whether the statements $(P \wedge Q) \vee R$ and $(P \wedge R) \vee (Q \wedge R)$ are logically equivalent.

P	Q	R	$P\wedge Q$	* $(P\wedge Q)\vee R$	$P\wedge R$	$Q\wedge R$	* $(P\wedge R)\vee(Q\wedge R)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	F	F
T	F	T	F	T	T	F	T
T	F	F	F	F	F	F	F
F	T	T	F	T	F	T	T
F	T	F	F	F	F	F	F
F	F	T	F	T	F	F	F
F	F	F	F	F	F	F	F

$(P\wedge Q)\vee R$ and $(P\wedge R)\vee(Q\wedge R)$ are not logically equivalent because they do not have the same truth values under all circumstances.

Excellent!

5. $\sqrt{6}$ is irrational.

First, note that for $a \in \mathbb{Z}$, $a^2 \equiv_6 0 \Rightarrow a \equiv_6 0$.*

Now s'pose $\sqrt{6}$ is rational, so $\sqrt{6} = \frac{p}{q}$ for integers p and q with no common factors. Then squaring both sides gives $6 = \frac{p^2}{q^2}$, or $6q^2 = p^2$. But then the left side is congruent to 0 mod 6, so $p^2 \equiv_6 0$. But then $p \equiv_6 0$ too, so $p = 6r$ for some $r \in \mathbb{Z}$. Thus $6q^2 = (6r)^2$, or $6q^2 = 36r^2$, so $q^2 = 6r^2$. Then since $6r^2 \equiv_6 0$, $q^2 \equiv_6 0$, and thus $q \equiv_6 0$. But that means $q = 6s$ for some $s \in \mathbb{Z}$, so p and q do indeed share a factor of 6, a contradiction arrising from our supposition that $\sqrt{6}$ was rational. So $\sqrt{6}$ is irrational, as desired. □

*It's easiest to show the contrapositive of this:

$a \equiv_6 1 \Rightarrow a^2 \equiv_6 1$

$a \equiv_6 2 \Rightarrow a^2 \equiv_6 4$

$a \equiv_6 3 \Rightarrow a^2 \equiv_6 3$

$a \equiv_6 4 \Rightarrow a^2 \equiv_6 2$

$a \equiv_6 5 \Rightarrow a^2 \equiv_6 1$

can all be confirmed by routine calculation.