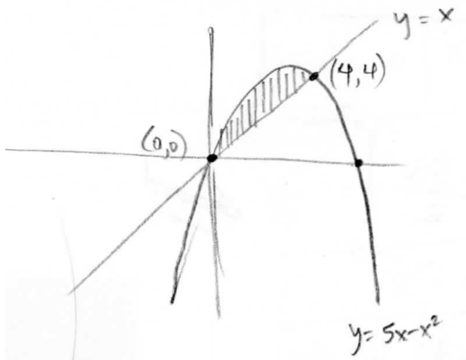
Exam 2a **Calculus 2** **3/15/19**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Set up an integral for the area of the region bounded between $y = x$ and $y = 5x - x^2$.

$x(5-x)$

$f(2) = 10 - 4 = 6$ ✓



$$\boxed{A = \int_0^4 (4x - x^2)\,dx}$$

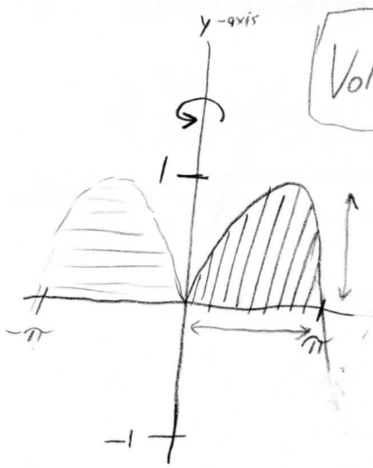
Good

$5x - x^2 = x$

$4x - x^2 = 0$

$x(4-x) = 0$ $\quad$ $x=0,\ x=4$

2. Set up an integral for the volume of the solid obtained when the region bounded between $y = \sin x$ and the x-axis is rotated around the y-axis.



$$\boxed{\text{Volume} = 2\pi \int_0^{\pi} \underbrace{x}_{\text{radius}}\,\underbrace{(\sin(x))}_{\text{height}}\,dx}$$

Nice!

3. A force of 5 pounds is required to hold a spring stretched 0.4 feet beyond its natural length. How much work (in foot-pounds) is done in stretching the spring from its natural length to 0.7 feet beyond its natural length?

$F = kx$

$5 = k(.4) \Rightarrow k = \frac{50}{4} = 12.5$

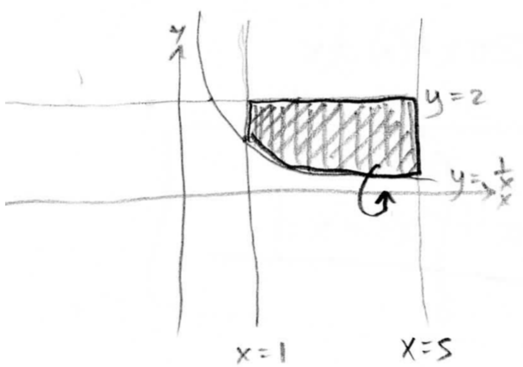
$W = Fd = \int F(x)\,dx$

$= \int_0^{.7} 12.5x\,dx = \left.\frac{12.5x^2}{2}\right]_0^{.7}$

Good

$\boxed{W = 3.0625 \text{ ft}\cdot\text{lbs.}}$

4. The region bounded between $y = 1/x, y = 2, x = 1$, and $x = 5$ is rotated around the x-axis. Set up an integral for the volume of the resulting solid.



Washer method:

$V = \int \pi \left(r_{outer}^2 - r_{inner}^2\right) dx$

$\boxed{V = \pi \int_1^5 \left[(2)^2 - \left(\frac{1}{x}\right)^2\right] dx}$

Good

5. Find a solution to the differential equation $\frac{dy}{dx} = \frac{x}{y}$ that satisfies the initial condition $y(0) = -3$.

$$\frac{dy}{dx} = \frac{x}{y}$$

Excellent!

$$\int y\,dy = \int x\,dx$$

separation of variables

$$\frac{y^2}{2} = \frac{x^2}{2} + C$$

$$y^2 = x^2 + C$$

$$y = \pm\sqrt{x^2 + C}$$

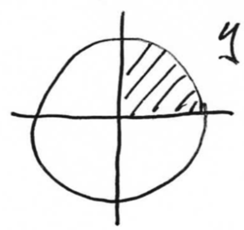
$$-3 = \pm\sqrt{(0)^2 + C}$$

$$= -\sqrt{C} \Rightarrow C = 9$$

$$\boxed{y = -\sqrt{x^2 + 9}}$$

6. Find $\bar{x}$, the x-coordinate of the centroid of the first-quadrant portion of a circle with radius 1 centered at the origin.

$$\boxed{\bar{x} = \frac{\int_a^b x \cdot f(x)\,dx}{\int_a^b f(x)\,dx}}$$



$y = \sqrt{1 - x^2}$

$$\bar{x} = \frac{\int_0^1 x \cdot \sqrt{1 - x^2}\,dx}{\int_0^1 \sqrt{1 - x^2}\,dx}$$

The denominator is the area of one quarter of a circle with radius 1, so $\frac{1}{4} \cdot \pi$

$$\text{So } \bar{x} = \frac{1/3}{\pi/4} = \frac{1}{3} \cdot \frac{4}{\pi} = \frac{4}{3\pi}$$

For the numerator, let $u = 1 - x^2$

$$\frac{du}{dx} = -2x$$

$$\frac{du}{-2x} = dx$$

$$\text{So } \int_0^1 x\sqrt{1 - x^2}\,dx = \int_1^0 \cancel{x} \cdot u^{1/2} \cdot \frac{du}{-2\cancel{x}}$$

$$= \frac{-1}{2}\int_1^0 u^{1/2}\,du$$

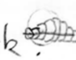
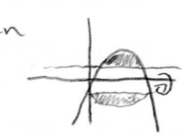
$$= \frac{-1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_1^0$$

$$= 0 - -\frac{1}{3} \cdot 1^{3/2}$$

$$= \frac{1}{3}$$

7. Bunny is a Calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, Calc is so impossible! It's totally unfair! And my teacher has such an accent! Like, there's the disco method and the washo method, I guess? And they're kinda alike but they're not? I'm so confused!"

Help Bunny out by explaining similarities and differences between the disc and washer methods.

Essentially Bunny, washers are discs, but with a hollow inside. The formula $\int_a^b \pi(f(x))^2 - \pi(g(x))^2\,dx$ can be used for both. For example, if I said find the volume of the solid formed when $y=\sin x$ is rotated around the x axis from $x=0$ to $x=\pi$, the graph would look like . Here, a cross section would look like a disk . The equation works because $g(x)=0$. A washer equation would look more like the volume of the solid formed by rotating the region bounded bet. $y=3x-x^2$ and $y=1$ around the x axis. Then  the cross section would look like a washer  with a hollow inside. The equation would have two functions because you have to subtract the inner area with an inner radius to get rid of the hollow area. So, almost the same, but not quite

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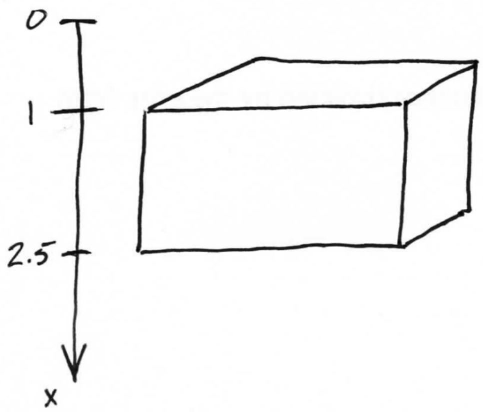
Great

8. An aquarium 3m long, 2m wide, and 1.5 m deep is full of water. Set up an integral for the amount of work required to pump all of the water in the tank up to a height 1 m above the top of the tank.



Area of a Slice = $2\,m \cdot 3\,m$

Volume of a Slice = $6\,m^2 \cdot \Delta x\,m$

Mass of a Slice = $6\Delta x\,m^3 \cdot 1000\,kg/m^3$

Force for a Slice = $6000\Delta x\,kg \cdot 9.8\,m/s^2$

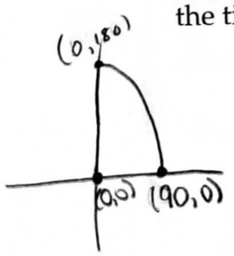
Work for a Slice = $58800\,N \cdot x\,m$

$$\text{Total Work} = \int_1^{2.5} 58800x\,dx \ J$$

9. [Stewart] A hawk flying 15 m/s at an altitude of 180 m accidentally drops its prey. The parabolic trajectory of the falling prey is described by the equation

$$y = 180 - \frac{x^2}{45}$$

until it hits the ground. Set up an integral for the distance traveled by the prey from the time it's dropped until it hits the ground.



$$\text{Arc Length} = \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

$$= \int_a^b \sqrt{1+\left(-\tfrac{2}{45}x\right)^2}\,dx$$

$y = 180 - \frac{1}{45}x^2$

$y' = -\frac{2}{45}x$

$0 = 180 - \frac{1}{45}x^2$

$\frac{1}{45}x^2 = 180$

$x^2 = 8100$

$x = 90$

$$\boxed{\text{Arc Length} = \int_0^{90} \sqrt{1+\left(-\tfrac{2}{45}x\right)^2}\,dx}$$

Excellent!

10. The solid formed by rotating the curve

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

where $a > b$, about the y-axis is called an oblate spheroid, and is essentially the shape of most planets (including Earth). Find the surface area of this solid.

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2}$$

$$y = b\sqrt{1 - \frac{x^2}{a^2}}$$

$$y' = b\frac{1}{2}\left(1 - \frac{x^2}{a^2}\right)^{-1/2} \cdot \frac{-2x}{a^2}$$

$$= -\frac{bx}{a^2}\left(\frac{a^2-x^2}{a^2}\right)^{-1/2}$$

$$= -\frac{bx}{a^2} \cdot \frac{a}{\sqrt{a^2-x^2}}$$

$$= -\frac{bx}{a} \cdot \frac{1}{\sqrt{a^2-x^2}}$$

$$\text{Area} = \int_{-a}^{a} 2\pi \underbrace{\left(b\sqrt{1-\frac{x^2}{a^2}}\right)}_{f(x)} \underbrace{\sqrt{1 + \left(-\frac{bx}{a} \cdot \frac{1}{\sqrt{a^2-x^2}}\right)^2}}_{1+[f'(x)]^2} \, dx$$

$$= 2\int_0^a 2\pi b \frac{\sqrt{a^2-x^2}}{a} \sqrt{1 + \frac{b^2x^2}{a^2} \frac{1}{a^2-x^2}} \, dx$$

$$= \frac{4\pi b}{a} \int_0^a \sqrt{a^2-x^2} \sqrt{\frac{a^2(a^2-x^2)}{a^2(a^2-x^2)} + \frac{b^2x^2}{a^2(a^2-x^2)}} \, dx$$

$$= \frac{4\pi b}{a^2} \int_0^a \sqrt{a^4 - a^2x^2 + b^2x^2} \, dx$$

$$= \frac{4\pi b}{a^2} \int_0^a \sqrt{a^4 - (a^2-b^2)x^2} \, dx$$

Now let $u = \sqrt{a^2-b^2}\,x$

$$\frac{du}{dx} = \sqrt{a^2-b^2}$$

$$dx = \frac{du}{\sqrt{a^2-b^2}}$$

$$= \frac{4\pi b}{a^2} \int_0^{a\sqrt{a^2-b^2}} \sqrt{a^4 - u^2} \, \frac{du}{\sqrt{a^2-b^2}}$$

$$= \frac{4\pi b}{a^2\sqrt{a^2-b^2}} \left[\frac{u}{2}\sqrt{a^4-u^2} + \frac{a^4}{2}\sin^{-1}\left(\frac{u}{a^2}\right)\right]_0^{a\sqrt{a^2-b^2}}$$

$$= \frac{4\pi b}{a^2\sqrt{a^2-b^2}} \left[\frac{a\sqrt{a^2-b^2}}{2}\; a^4 - a^2(a^2-b^2) + \frac{a^4}{2}\sin^{-1}\frac{\sqrt{a^2-b^2}}{a}\right]$$

$$= 2\pi \; b^2 + \frac{a^2 b \sin^{-1}\frac{\sqrt{a^2-b^2}}{a}}{\sqrt{a^2-b^2}}$$