Exam 3a **Calculus 2** **4/5/19**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Find the sum of the geometric series

$$\frac{1}{2}-\frac{1}{4}+\frac{1}{8}-\frac{1}{16}+\frac{1}{32}-\frac{1}{64}+\frac{1}{128}-\frac{1}{256}+\ldots$$

$\Sigma \to \frac{a}{1-r}$

$a = \frac{1}{2}$

$r = -\frac{1}{2}$

Good

$\boxed{\text{Sum of the geometric series} = 1/3}$

$$\frac{\left(\frac{1}{2}\right)}{1-\left(-\frac{1}{2}\right)} \to \frac{\left(\frac{1}{2}\right)}{\frac{2}{2}+\frac{1}{2}} \to \frac{\left(\frac{1}{2}\right)}{\left(\frac{3}{2}\right)} = \frac{1}{2}\cdot\frac{2}{3} = \frac{2}{6} = \frac{1}{3}$$

2. Find the first 3 partial sums of the series

$$\sum_{n=1}^{\infty}\frac{(-1)^n}{(2n-1)!}$$

$$S_1 = \frac{(-1)^1}{(2-1)!} = \boxed{-1}$$

$$S_2 = S_1 + \frac{(-1)^2}{(4-1)!} = -1 + \frac{1}{6} = \boxed{-\frac{5}{6}}$$

$$S_3 = S_2 + \frac{(-1)^3}{(6-1)!} = -\frac{5}{6} + \frac{-1}{5\cdot4\cdot6}$$

$$= \frac{-5}{6} + \frac{-1}{120} = \frac{-100}{120} + \frac{-1}{120}$$

$$= \boxed{\frac{-101}{120}}$$

Great

3. Find the 5th degree MacLaurin polynomial for $f(x) = e^x$.

$$p(x) = f(0) + f'(0)\cdot x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \frac{f^{(4)}(0)x^4}{4!} + \frac{f^{(5)}(0)x^5}{5!}$$

$f(x) = e^x \rightarrow f(0) = e^0 = 1$

$f'(x) = e^x$ $\quad f'(0) = 1$

$f''(x) = e^x$ $\quad f''(0) = 1$

$f'''(x) = e^x$

$f^{(4)}(x) = e^x$ $\quad$ and so on

$f^{(5)}(x) = e^x$

Excellent!

$$\boxed{p(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}}$$

for e^x

4. Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n+1}}$ converges or diverges.

Integral Test:

Since $f(x) = \frac{1}{\sqrt{x+1}}$ is continuous, positive, and decreasing on $[0, \infty)$

Let $u = x+1$

$du = dx$

$$\int_0^{\infty} \frac{1}{\sqrt{x+1}}\,dx = \lim_{b\to\infty} \int_0^b \frac{1}{\sqrt{x+1}}\,dx$$

$$= \lim_{b\to\infty} \int_{0=x}^{b=x} \frac{1}{\sqrt{u}}\,du \qquad \frac{1}{\sqrt{u}} = u^{-\frac{1}{2}}$$

$$= \lim_{b\to\infty} \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right]_{0=x}^{b=x}$$

Excellent!

$$= \lim_{b\to\infty} \left[2(x+1)^{\frac{1}{2}} \right]_0^b = \lim_{b\to\infty} \left(2(b+1)^{\frac{1}{2}} - 2(1)^{\frac{1}{2}} \right)$$

$$= \infty$$

Since the integral diverges, the series diverges as well.

5. Determine whether the series $\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}$ converges or diverges.

Ratio Test:
$$\lim_{n\to\infty} \left| \frac{\frac{1}{(2(n+1)+1)!}}{\frac{1}{(2n+1)!}} \right| = \lim_{n\to\infty} \left| \frac{(2n+1)!}{(2n+3)!} \right|$$

$$= \lim_{n\to\infty} \left| \frac{1}{(2n+2)(2n+3)} \right| = 0. \quad \text{Since } 0 < 1$$

the series converges absolutely.

Nice

6. Determine whether the series $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ converges or diverges.

AST

1. Alt series ✓
2. $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓
3. $\frac{d}{dx}\frac{1}{n} = \frac{d}{dx} n^{-1} = -n^{-2} = \frac{-1}{n^2}$ ∴ Decreasing ✓

By AST, we know that

$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$ is converging

Excellent

7. Biff is a Calculus student at Enormous State University, and he's having some trouble. Biff says "Crap, Calc isn't math! I mean, I can work out answers and stuff, but now they want reasons. How is that math? Like, on our last exam, there was this one where we were supposed to multiple guess, like, which series you'd know diverged by the test for divergence. How stupid is that? If the answer isn't, like, 7 or something, then it's not math. I think this is really philosophy or something."

   Help Biff out by giving an example of a series that the Test for Divergence tells us is divergent, and explain why.

For a series like $\sum_{n=0}^{\infty} 7^n$, we know that this diverges b/c $\lim_{n\to\infty} 7^n \neq 0$ so we know it is divergent. This is b/c it continuously increases to ∞ $\therefore$ Diverges.

This is different for things like $\sum_{n=0}^{\infty} \frac{1}{10^n}$ b/c $\lim_{n\to\infty} \frac{1}{10^n} = 0$, but we cannot make conclusions that it converges via the divergence test since it cannot do that. we would have to use a different test to prove convergence.

Great.

8. Find the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{x^n}{n}$$

Ratio Test:

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{n+1}}{\frac{x^n}{n}} \right| = \lim_{n\to\infty} \left| \frac{\cancel{x^{n+1}}\, n}{(n+1)\cancel{x^n}} \right|$$

$$= \lim_{n\to\infty} \left| \frac{nx}{n+1} \right|$$ → this is an indeterminate form, so use L'Hopital's Rule

taking the derivative of top + bottom with respect to n →

$$\overset{L'H}{=} \lim_{n\to\infty} \left| \frac{x}{1} \right| = |x| = L.$$

We know the series converges when $L < 1$,

so if $L = |x|$, then the series

converges when $|x| < 1$.

$$\Rightarrow -1 < x < 1$$

$\boxed{\text{radius} = 1}$

Well done!

9. Find the interval of convergence of the power series

$$\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$$

Ratio Test:

$$\lim_{n\to\infty}\left|\frac{\frac{\cancel{(-1)^{n+1}}x^{2n+2}}{(2n+2)!}}{\frac{\cancel{(-1)^n}x^{2n}}{(2n)!}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{(-1)x^{\cancel{2n+2}}(2n)!}{\cancel{x^{2n}}(2n+2)!}\right|$$

the -1 disappears because of the absolute value sign and is never seen again.

$$= \lim_{n\to\infty}\left|\frac{x^2}{(2n+1)(2n+2)}\right| = 0 \text{ for all } x.$$

The absolute value has connections in high places and is never tried in court or even accused. :(

Therefore, the interval of convergence for x is $\boxed{(-\infty, \infty)}$.

Excellent!

10. Use a power series with at least 3 nonzero terms to approximate

$$\int_0^{0.1} \cos(x^2)\, dx$$

I know that

$$\cos(x) \approx 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!}$$

$$\text{So } \cos(x^2) \approx 1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!}$$

Polynomials are easy to integrate, so:

$$\int_0^{.1} \cos(x^2)dx \approx \int_0^{.1} \left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} - \frac{x^{12}}{6!}\right)dx$$

which equals $\left[x - \frac{x^5}{5 \cdot 2} + \frac{x^9}{9 \cdot 24} - \frac{x^{13}}{13 \cdot 6!} \right]_0^{.1}$

That is,

$$\left((.1) - \frac{(.1)^5}{10} + \frac{(.1)^9}{216} - \frac{(.1)^{13}}{9{,}360} \right) - (0)$$

$$\boxed{\approx .099999}$$

Excellent!