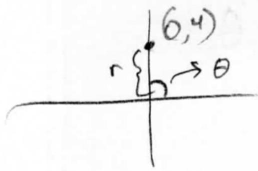
Exam 4 **Calculus 2** **4/26/19**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Convert the point with rectangular coordinates $(0,4)$ to polar coordinates (r,θ).

$r = \sqrt{x^2+y^2} = \sqrt{0^2+4^2} = 4$



$r = 4$
$\theta = \frac{\pi}{2}$

$\boxed{\left(4, \frac{\pi}{2}\right)}$

Good

2. Consider the curve defined by the parametric equations $x(t) = t^2 - 1$ and $y(t) = t^2 - t$. Find the slope of this curve at the point $(0,0)$.

$\frac{dy}{dx} = \frac{2t-1}{2t}$

$0 = t^2 - 1$
$1 = t^2$
$t = \pm 1$

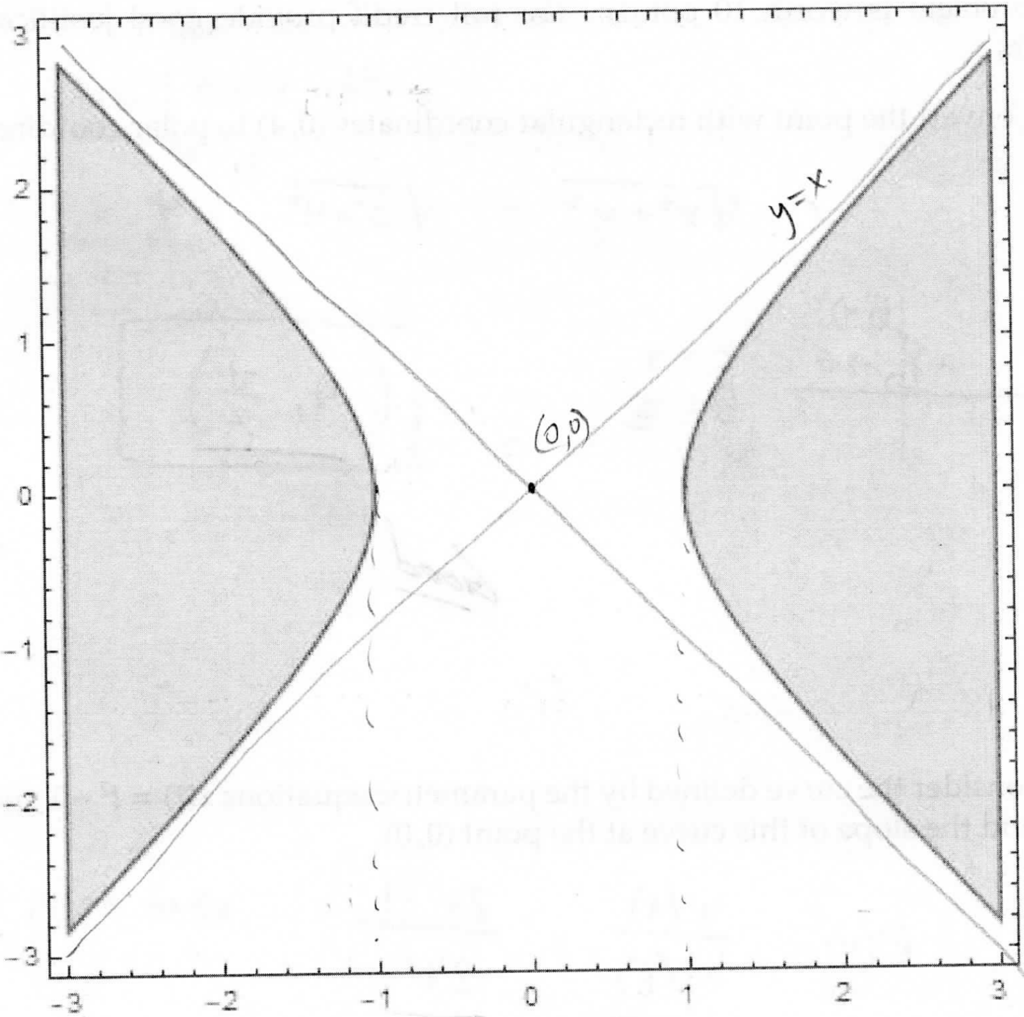
$0 = t^2 - t$
$0 = t(t-1)$
$t = 0, 1$

$t = 1$

$= \frac{2(1)-1}{2(1)} = \boxed{\frac{1}{2} = \text{slope}}$

Good

3. Find an equation for the hyporbola shown:



center @ (0,0) : $\frac{(x-0)^2}{1} - \frac{(y-0)^2}{1} = 1$

vertices @ (±1, 0) and asymptote y=x.

Good : $\boxed{x^2 - y^2 = 1}$

check $y^2 = \sqrt{x^2-1}$

4. Consider the curve defined by the parametric equations $x = t^3 - 7t$ and $y = 8t^2$. Set up an integral for the length of the loop of this curve.

Length $= \int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$

$x = t^3 - 7t$ $\quad y = 8t^2$

$\downarrow$ derivative $\downarrow$

$3t^2 - 7 \quad 16t$

Intersections

$t^3 - 7t = 0$

$\downarrow$

$\frac{\cancel{t}(t^2 - 7)}{\cancel{t}} = \frac{0}{t}$

$t^2 - 7 = 0$
$\quad +7 \quad +7$

$\sqrt{t^2} = 7$

$t = \pm\sqrt{7}$

$$\boxed{\int_{-\sqrt{7}}^{\sqrt{7}} \sqrt{[3t^2 - 7]^2 + [16t]^2}\, dt}$$

Great

5. Set up an integral for the area bounded by the curve with polar equation $r = 2\sin(3t)$.

$$A = \int_{\alpha}^{\beta} \frac{1}{2}\left[r(\theta)\right]^2 d\theta$$

$0 = 2\sin(3t)$

$0 = \sin(3t)$

$t = 0$ or $t = \pi/3$

1 leaf $= \displaystyle\int_0^{\pi/3} \frac{1}{2}\left[2\sin(3t)\right]^2 dt$

Area for all 3 leafs

$$\boxed{= \frac{3}{2}\int_0^{\pi/3} \left[2\sin(3t)\right]^2 dt}$$

Good

6. Identify the graph of $y^2 - 8y = 16x^2$ as a parabola, hyperbola, or ellipse, give coordinates of its vertices, and sketch a decent graph.

3 things

$y^2 - 8y = 16x^2$ not a parabola

$\cancel{y(y-8) = (4x)^2}$

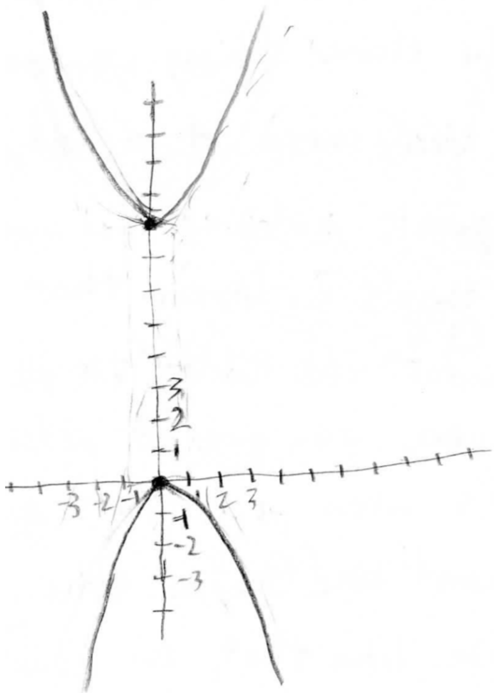
$y^2 - 8y + 16 = 16x^2 + 16$

$\frac{(y-4)^2}{1^2} - 16x^2 = 16$

$\frac{(y-4)^2}{4^2} - \frac{x^2}{1^2} = 1$

Excellent

$\boxed{\text{hyperbola}}$



vertices: $(0,0)$ and $(0,8)$

7. Bunny is a Calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod. Ohmygod. This is just so confusing. I got all the answers to our homework problems from the online thingy, right? Where they have, like, all the answers for problems in the whole book, right? But then on our exam nothing went right, which is totally unfair, right? But so what I really need to know, like for the final, is for the rose things, like the *r* is cos 5 theta or 6 theta or something, how do you even know if your limits are to pi or to 2pi? Because in the online answers, it's like, both ways, right?"

$r = \cos 5\theta$
$r = \cos 6\theta$

Help Bunny out by explaining how to tell whether a limit of π or 2π is appropriate for problems of this sort.

Problems like these that result in rose-shaped graphs two different outcomes depending on whether they have an even value times theta, or an odd one, where an odd value $(x\theta)$ will result in in that number of "petals" while an even value will result in twice as many.

The odd value also travels once from starting point back to itself when $\theta = \pi$ and twice while $\theta = 2\pi$, whereas the even value will only travel half way when $\theta = \pi$ and all the way when $\theta = 2\pi$.

So given this, if some value of x exists where $r = \cos(x\theta)$, if x is odd, the best limit of θ is π and if it is even, the best limit of θ is 2π.

Good

8. Find the area of the region which is inside the polar curve

$$r = 4\cos(\theta)$$

and outside the curve

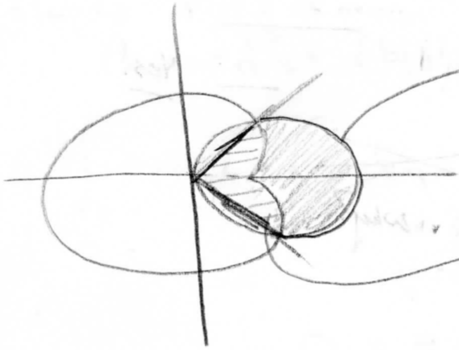
$$r = 3 - 2\cos(\theta)$$

$4\cos\theta = 3 - 2\cos\theta$

$6\cos\theta = 3$

$\cos\theta = \frac{1}{2}$

$\theta = \pi/3$



$$\frac{1}{2}\int_{-\pi/3}^{\pi/3} [4\cos\theta]^2 \, d\theta \approx 11.84168202$$

$$\frac{1}{2}\int_{-\frac{\pi}{3}}^{\frac{\pi}{3}} [3 - 2\cos\theta]^2 \, d\theta$$

≈ 1.992893622

10

$11.84168202 - 1.992893622$

by calculator

$\boxed{\approx 9.848788398}$

9. Find the exact (x, y) coordinates of the highest point(s) on the curve with polar equation $r = 1 - 2\sin\theta$.

that means $\underline{\text{slope} = 0}$ there

Slope of a polar equation $= \dfrac{\frac{dr}{d\theta}\sin\theta + r(\theta)\cos\theta}{\frac{dr}{d\theta}\cos\theta - r(\theta)\sin\theta}$ ← got this by applying Product Rule to $r(\theta)\sin\theta$ and $r(\theta)\cos\theta$ for num + denom respectively

$r' = -2\cos\theta$

$$= \frac{(-2\cos\theta)\sin\theta + (1-2\sin\theta)\cos\theta}{\text{blah blah blah}}$$

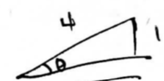
→ we don't care about this because we just need the $\underline{\text{numerator}} = 0$

So:

$$0 = -2\cos\theta\sin\theta + \cos\theta - 2\sin\theta\cos\theta$$

$$0 = \cos\theta - 4\cos\theta\sin\theta$$

$$0 = \cos\theta\,(1 - 4\sin\theta)$$



$\sqrt{4^2-1^2}$
$= \sqrt{15}$

$\cos\theta = 0$
$\theta = \frac{\pi}{2}$

$1 - 4\sin\theta = 0$
$\sin\theta = \frac{1}{4}$

$\cos\theta = \frac{\sqrt{15}}{4}$

$\theta = \sin^{-1}\left(\frac{1}{4}\right)$

$x = r\cos\theta = \left(1-2\left(\frac{1}{4}\right)\right)\cos\left(\sin^{-1}\left(\frac{1}{4}\right)\right)$

$y = r\sin\theta = \left(1-\frac{1}{2}\right)\sin\left(\sin^{-1}\left(\frac{1}{4}\right)\right)$

$\left(\frac{1}{2}\left(\frac{\sqrt{15}}{4}\right), \left(\frac{1}{2}\right)\left(\frac{1}{4}\right)\right)$

$\boxed{(x, y) = }$

$= \boxed{\left(\frac{\sqrt{15}}{8}, \frac{1}{8}\right)}$ and $\boxed{\left(-\frac{\sqrt{15}}{8}, \frac{1}{8}\right)}$

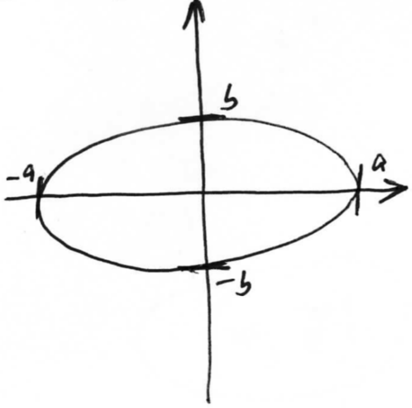
Excellent

10. An ellipse can be expressed with the parametric equations

$$x(t) = a\cos t$$

$$y(t) = b\sin t$$

Set up an integral for the area bounded by the ellipse.



$$\text{Area} \overset{?}{=} \int_0^{2\pi} (b\sin t)(-a\sin t)\,dt$$

$$= -ab\int_0^{2\pi} \sin^2 t\,dt$$

But that's right-to-left in quadrants I and II, then under the axis in quadrants III and IV, so each part gets us the negative of the actual area, and we want

$$ab\int_0^{2\pi} \sin^2 t\,dt$$