Exam 3 **Foundations of Advanced Math** **3/29/19**

1. The sum of two bounded functions, both with domain $\mathbb{R}$, is bounded.

let f be bounded such that

$|f(x)| \leq M_1$

let g be bounded such that

$|g(x)| \leq M_2$

$|f(x)| + |g(x)| \leq M_1 + M_2$

by the triangle inequality,

$|f(x) + g(x)| \leq |f(x)| + |g(x)|$

so by the transitive property,

$|f+g(x)| \leq M_1 + M_2$

Nice!

and thus $f+g$ is bounded

by def. $\square$

2. (a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is an even function. Then $g(x) = [f(x)]^2$ is also an even function.

Well, we know $\forall x \in \mathbb{R}$, $f(-x) = f(x)$. $*$

Then $g(-x) = [f(-x)]^2$ by def. of g

$= [f(x)]^2$ by $*$

$= g(x)$ by def. of g

So since $\forall x \in \mathbb{R}$, $g(-x) = g(x)$, then g is even by definition. $\square$

(b) Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function for which $g(x) = [f(x)]^2$ is an even function. Then f is also an even function.

Consider $f(x) = x$.

Then $g(x) = [f(x)]^2 = x^2$ is even by previous work.

But $f(x) = x$ is not even, since $f(2) = 2$ and $f(-2) = -2$,

but $2 \neq -2$.

So we have a counterexample and the proposition is not always true. $\square$

3. If $f : A \to B$ and $g : B \to C$ are surjective functions, then $g \circ f$ is surjective.

As g is surjective, $\forall c \in C$, $\exists b \in B$ where $g(b) = c$

As f is surjective for this b there exists an $a \in A$ where $f(a) = b$

Therefore $(g \circ f)(a) = g(f(a)) = g(b) = c$

and $\forall c \in C$, an $a \in A$ can be found.

$\therefore$ when g & f are surjective $g \circ f$ is surjective

Good.

4. If $f : A \to B$ and $g : B \to C$ are injective functions, then $g \circ f$ is injective.

let $g(f(a_1)) = g(f(a_2))$, then since g is injective we know $f(a_1) = f(a_2)$ and since f is injective, we know $a_1 = a_2$. Thus $g(f(a_1)) = g(f(a_2)) \Rightarrow a_1 = a_2$ and therefore $g(f(a))$ is injective, which is $(g \circ f)(a)$, and thus $g \circ f$ is injective. $\square$

Great

5. (a) The set of natural numbers is equipollent to the set of even natural numbers.

Two sets are equipollent if there exists a bijection from one set to the other.

Take $f: \mathbb{N} \to$ even naturals, where $f(n) = 2n$.

Good

(i.) $f(n_1) = f(n_2) \Rightarrow 2n_1 = 2n_2 \Rightarrow n_1 = n_2$

Therefore f is injective.

(ii) For any even natural number, b, $\exists\, n \in \mathbb{N}$ such that $f(n) = b$.

$b = 2n \Rightarrow \frac{b}{2} = n.$

Therefore f is surjective. By (i) and (ii) we have shown that a bijection exists $\therefore$ $\mathbb{N}$ and the even naturals are equipollent. $\blacksquare$

(b) The set $\{n | n \in \mathbb{N} \wedge n \geq 58\}$ is countable.

A set is countable if it is equipollent with some subset of $\mathbb{N}$, that is, there exists a bijection from that set to N_1, a subset of $\mathbb{N}$.

Take $f: \{n | n \in \mathbb{N} \wedge n \geq 58\} \to N_1$ where

$N_1 = \{n | n \in \mathbb{N} \wedge n \geq 58\}$ and $f(x) = x$,

Since the identity function is a bijection on any set, and the set in question is itself a subset of $\mathbb{N}$, we can say it is countable.

Excellent