Examlet 4b **Foundations of Advanced Math** **4/16/21**

1. Consider the relation $\sim$ on $\mathbb{Z}$ defined by $x \sim y \Leftrightarrow |x-y| > 3$.

   (a) Find 3 elements of $\mathbb{Z}$ that are related to 2.

   $\underline{6} \sim 2$

   $\underline{7} \sim 2$

   $\underline{8} \sim 2$

   (b) Find 3 elements of $\mathbb{Z}$ that are not related to 2.

   $\underline{0} \nsim 2$

   $\underline{1} \nsim 2$

   $\underline{2} \nsim 2$

   (c) Determine whether $\sim$ is an equivalence relation.

   Reflexive: To be reflexive, $\forall x \in \mathbb{Z}$, $x \sim x$. So, it must be true that $|x-x| > 3$. For any x, $x-x=0$, and $|0| = 0$. It is not true that $0 > 3$, so the relation $\sim$ is not reflexive. It cannot be an equivalence relation because it is not reflexive, symmetric, and transitive.

Good

2. Let $S = \{a, b, c, d, e\}$, and let $\sim = \{(a,a), (b,b), (b,d), (b,e), (c,c), (d,b), (d,d), (d,e), (e,b), (e,d), (e,e)\}$
.

(a) Give the equivalence classes of $\sim$.

$[a] = \{a\}$

$[b] = \{b, d, e\}$

$[c] = \{c\}$

$[d] = \{d, b, e\}$

$[e] = \{e, b, d\}$

Great

(b) Give the partition associated with $\sim$.

$\Pi = \{\{a\}, \{b, d, e\}, \{c\}\}$

3. Let S be a set and Π a partition of S. Let $\sim$ be a relation on S defined by $a \sim b \Leftrightarrow \exists P \in \Pi$ for which $a, b \in P$.

(a) Show $\sim$ is a reflexive relation.

Knowing that Π is a partition of S, let $a \in S$ and therefore $a \in P$ for $P \in \Pi$ since the union of all sets in Π is all of S. Since $a \in P$ and $a \in P$, $a \sim a$ and is therefore reflexive.

(b) Show $\sim$ is a symmetric relation.

Well, suppose that $a \sim b$ for which $a, b \in P$ if $\exists P \in \Pi$. Then we can say that $b, a \in P$, so $b \sim a$. So $a \sim b \Rightarrow b \sim a$, and is therefore symmetric.

(c) Show $\sim$ is a transitive relation.

Well suppose $a \sim b$ and $b \sim c$ if $P_1 \in \Pi$ for $a, b \in P_1$ and $P_2 \in \Pi$ for $b, c \in P_2$. But then $b \in P_1 \cap P_2$ and $P_1 \cap P_2 \neq \emptyset$. Since Π is pairwise disjoint, $P_1 = P_2$, so then $a, b, c \in P_1 = P_2$, so $a \sim c$.
So $a \sim b \wedge b \sim c \Rightarrow a \sim c$, and is therefore transitive.

Good

4. Let S be a set and define a relation on the subsets of S by saying $T \sim U$ iff there exists a bijection from T to U.

(a) Determine whether $\sim$ is a reflexive relation, and why.

$T \sim T$ iff there exists a bijection from T to T. Consider the identity function. For every element in T, the identity function outputs that same element. So, $T \sim T$ and $\sim$ is reflexive. $\square$

Good

(b) Determine whether $\sim$ is a symmetric relation, and why.

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If $T \sim U$, then there exists some bijection from T to U. We know that every bijection has an inverse bijection, so there exists a bijection from U to T as well. So, $U \sim T$ and $\sim$ is symmetric. $\square$

Good

(c) Determine whether $\sim$ is a transitive relation, and why.

If $T \sim U$ and $U \sim V$, then there exists some bijection f from T to U and some bijection g from U to V. Then, the composition of f and g gives us a bijection from T to V. So, $T \sim V$ and $\sim$ is transitive. $\square$ Great

5. In any graph, the number of vertices of odd degree is even.

Suppose false, and there were an odd number of odd degree vertices. The sum of degrees of all even degree vertices must be even. The sum of degrees in an odd number of odd vertices must be odd. So adding those together, the total degree sum must be odd. However, we know the total degree sum must be even. So the total number of odd degree vertices must be even instead.

Good