Exam 1 **Calculus 2** **2/2/22**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Evaluate

$$\int \frac{1}{3x+2}\,dx$$

1)

$u = 3x+2$

$\frac{du}{dx} = 3$

$dx = \frac{du}{3}$

$\int \frac{1}{3u} \cdot du$

$\frac{1}{3} \int \frac{1}{u}\, du$

$\frac{1}{3} \ln|u| + c$

Great

$\frac{1}{3} \ln|3x+2| + c$

2. Evaluate

$$\int x \sin x \, dx$$

$\int u \cdot dv = uv - \int u' \cdot v$

$\int x \sin x \, dx$

$u = x$ $\quad v = -\cos x$

$u' = 1$ $\quad v' = \sin x$

$= -x\cos x - \int 1 \cdot -\cos x$

constant

$= -x\cos x + \int \cos x$

Great!

$= -x\cos x + \sin x$

$= \sin x - x\cos x + C$

3. Evaluate

$$\int \cos^4\theta \sin\theta \, d\theta$$

$$\int \cos^4\theta \sin\theta \, d\theta = \int u^4 \cancel{\sin\theta} \frac{du}{-\cancel{\sin\theta}} = -\int u^4 du$$

$u = \cos\theta$

$\frac{du}{dx} = -\sin\theta$

$dx = \frac{du}{-\sin\theta}$

$$-\int u^4 du = -\frac{1}{5}u^5 + C = -\frac{1}{5}\cos^5\theta + C$$

$$\int \cos^4\theta \sin\theta \, d\theta = -\frac{1}{5}\cos^5\theta + C$$

Great!

4. Evaluate

$$\int \frac{5x}{(x+3)(x-2)}\,dx$$

$$\frac{5x}{(x+3)(x-2)} = \frac{A}{x+3} + \frac{B}{(x-2)}$$

multiply both sides by denom

$$5x = A(x-2) + B(x+3)$$

$$0 + 5x = Ax - 2A + Bx + 3B$$

$$5 = (A+B)x$$

$$0 = -2A + 3B$$

$$\int \frac{5x}{(x+3)(x-2)}\,dx = \int \frac{3}{(x+3)} + \frac{2}{(x-2)}\,dx$$

u-subs for each

$$= 3\ln|x+3| + 2\ln|x-2| + C$$

$$5 = A + B$$
$$-B$$
$$5 - B = A$$

$$0 = -2(5-B) + 3B$$
$$= -10 + 2B + 3B$$
$$\frac{10}{5} = \frac{5B}{5}$$
$$\boxed{2 = B}$$

$$A = 5 - 2$$
$$\boxed{A = 3}$$

Nice!

5. Evaluate

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx$$

Let $u = 1-x^2 \Rightarrow x^2 = 1-u$

$\frac{du}{dx} = -2x$

$\frac{du}{-2x} = dx$

$$\int \frac{x^3}{\sqrt{1-x^2}}\,dx = \int \frac{x^3}{u^{1/2}} \cdot \frac{du}{-2x}$$

$$= \frac{-1}{2}\int x^2 \cdot u^{-1/2}\,du$$

$$= \frac{-1}{2}\int (1-u)\cdot u^{-1/2}\,du$$

$$= \frac{-1}{2}\int \left(u^{-1/2} - u^{1/2}\right)du$$

$$= \frac{-1}{2}\left(2\cdot u^{1/2} - \frac{2}{3}u^{3/2}\right) + C$$

$$= -(1-x^2)^{1/2} + \frac{1}{3}(1-x^2)^{3/2} + C$$

or

$$= \sqrt{1-x^2}\left(-1 + \frac{1}{3}(1-x^2)\right) + C$$

$$= -\sqrt{1-x^2}\left(\frac{x^2+2}{3}\right) + C$$

6. Evaluate

$$\int_0^{\pi/2} \cos^2\theta\, d\theta$$

$$\int_0^{\pi/2} \cos^2\theta\, d\theta = \int_0^{\pi/2} \frac{1}{2}\left(1+\cos(2\theta)\right)d\theta = \frac{1}{2}\int_0^{\pi/2} 1+\cos(2\theta)\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} 1\, d\theta + \frac{1}{2}\int_0^{\pi/2} \cos(2\theta)\, d\theta$$

Let:

$u = 2\theta$

$\frac{du}{d\theta} = 2$

$\frac{du}{2} = d\theta$

$$= \frac{1}{2}\,\theta\Big|_0^{\pi/2} + \frac{1}{2}\int_{\theta=0}^{\theta=\pi/2} \frac{1}{2}\cos(u)\, du$$

$$= \frac{1}{2}(\pi/2 - 0) + \frac{1}{4}\int_{\theta=0}^{\theta=\pi/2} \cos(u)\, du$$

$\int \cos(u)\, du = \sin u + C$

$u = 2\theta$

$\sin(2\theta) + C$

$$= \frac{\pi}{4} + \frac{1}{4}\sin(2\theta)\Big|_0^{\pi/2}$$

$\sin(\pi) = 0$

$\sin(0) = 0$

$$= \frac{\pi}{4} + \frac{1}{4}(0-0) = \boxed{\pi/4}$$

Good

7. Bunny is a Calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, Calc is tough! This partial fractal stuff is totally impossible! I think it's totally random, like, sometimes you put just A or B and some other times you put, like, $Cx + D$, and there's totally no way to know which is which, it's just what the professor decides, right?"

   Help Bunny out by giving a good example to illustrate when to use which kind of numerator in a partial fractions decomposition. You do not need to carry out the decomposition, just give and explain to Bunny what form the decomposition should take.

The usage of A or B and Cx+D depends on the denominator of the function being integrated. Specifically, the power of x.

For example, let's take $\int \frac{1}{(x+5)(x-7)}\,dx$

Because the highest power of either of the factors in the denominator is x^1, we can split it up like so:

$$\frac{1}{(x+5)(x-7)} = \frac{A}{(x+5)} + \frac{B}{(x-7)}$$

However, if the power is higher, this changes.

For example, let's look at $\int \frac{3}{(x^2+8)(x-2)}\,dx$. Because we have a denominator factor with power x^2, that factor needs to have an x^1 in the numerator. So it looks like this:

$$\frac{3}{(x^2+8)(x-2)} = \frac{Ax+B}{(x^2+8)} + \frac{C}{(x-2)}$$

Great!

Note that only the term with (x^2+8) under it has the $Ax+B$.

Also, remember this does not apply to factors like $(x+3)^2$.

8. Evaluate

$$\int_0^5 \frac{1}{x-3}\,dx$$

Discontinuous when $x = 3$!

First look at $\int_3^5 \frac{1}{x-3}\,dx = \lim_{b \to 3^+} \int_b^5 \frac{1}{x-3}\,dx$

$$= \lim_{b \to 3^+} \ln|x-3| \Big|_b^5$$

$$= \lim_{b \to 3^+} \ln 2 - \ln|b-3|$$

But $\lim_{x \to 0^+} \ln x = -\infty$, so this diverges,

so $\int_0^5 \frac{1}{x-3}\,dx = \int_0^3 \frac{1}{x-3}\,dx + \int_3^5 \frac{1}{x-3}\,dx$

also diverges

9. Derive the reduction formula

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$$

Parts!

$u = (\ln x)^n \quad v = x$

$u' = n(\ln x)^{n-1} \cdot \frac{1}{x} \quad v' = 1$

$$\int (\ln x)^n dx = (\ln x)^n \cdot x - \int n(\ln x)^{n-1} \frac{1}{x} x\,dx$$

$$= x(\ln x)^n - n\int n(\ln x)^{n-1} dx$$

$\sin^2\theta + \cos^2\theta = 1$

$\tan^2\theta + 1 = \sec^2\theta$

10. Derive Line 39 from the Table of Integrals,

$$\int \sqrt{u^2 - a^2}\, du = \frac{u}{2}\sqrt{u^2 - a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2 - a^2}\right| + C$$

Trig. Sub.!

Let $u = a\sec\theta$

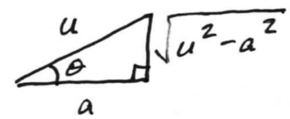
$\frac{du}{d\theta} = a\sec\theta\tan\theta$

$du = a\sec\theta\tan\theta\, d\theta$

If $u = a\sec\theta$

$\frac{u}{a} = \sec\theta$

$\cos\theta = \frac{a}{u}$



$$\int\sqrt{u^2-a^2}\,du = \int\sqrt{(a\sec\theta)^2 - a^2}\cdot a\sec\theta\tan\theta\,d\theta$$

$$= \int\sqrt{a^2\sec^2\theta - a^2}\cdot a\sec\theta\tan\theta\,d\theta$$

$$= a\int\sqrt{a^2(\sec^2\theta - 1)}\,\sec\theta\tan\theta\,d\theta$$

$$= a\int a\sqrt{\tan^2\theta}\,\sec\theta\tan\theta\,d\theta$$

$$= a^2\int \tan\theta\sec\theta\tan\theta\,d\theta$$

$$= a^2\int \tan^2\theta\sec\theta\,d\theta$$

$$= a^2\int (\sec^2\theta - 1)\sec\theta\,d\theta$$

$$= a^2\int (\sec^3\theta - \sec\theta)\,d\theta$$

$$= a^2\Big(\overbrace{\tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\ln|\sec\theta + \tan\theta|}^{\text{Line 71}} - \overbrace{\ln|\sec\theta + \tan\theta|}^{\text{Line 14}}\Big) + C$$

$$= a^2\left(\tfrac{1}{2}\sec\theta\tan\theta - \tfrac{1}{2}\ln|\sec\theta + \tan\theta|\right) + C$$

$$= \frac{a^2}{2}\cdot\frac{u}{a}\cdot\frac{\sqrt{u^2-a^2}}{a} - \frac{a^2}{2}\ln\left|\frac{u}{a} + \frac{\sqrt{u^2-a^2}}{a}\right| + C$$

$$= \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2-a^2}\right| - \ln|a| + C$$

$$= \frac{u}{2}\sqrt{u^2-a^2} - \frac{a^2}{2}\ln\left|u + \sqrt{u^2-a^2}\right| + C^*$$