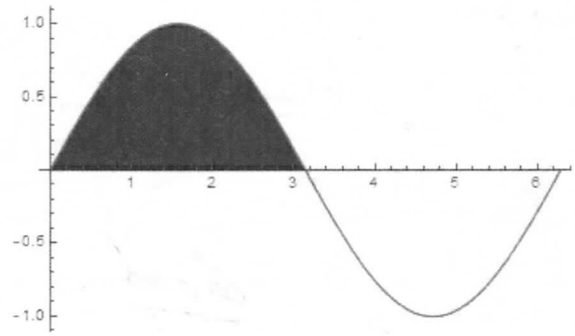
Exam 2 **Calculus 2** **3/2/22**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Set up an integral for the area of the shaded region shown between $y = \sin x$ and $y = 0$.



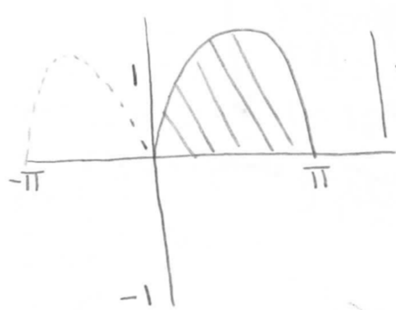
$\text{Area} = \int_a^b f(x)\,dx$

$\text{Area} = \int_0^{\pi} \sin x\,dx$

Good!

2. Set up an integral for the volume of the solid obtained when the region shown in Question 1 is rotated around the y-axis.

shell method



$h =$
$y = \sin x$

$\text{volume} = \int_a^b 2\pi\,(r)(\text{height})\,dx$

$\text{volume} = \int_0^{\pi} 2\pi\,(x)(\sin x)\,dx$

$\boxed{\text{volume} = 2\pi \int_0^{\pi} (x)(\sin x)\,dx}$

Great!

3. A force of 10 Newtons is required to hold a spring stretched 20 cm beyond its natural length. How much work (in Joules) is done in stretching the spring from 20 cm beyond natural length to 30 cm beyond natural length?

$W = f \cdot d \qquad f = k \cdot x$

$\frac{10N}{.20m} = \frac{k(.20m)}{.20m}$

$50 \frac{N}{m} = k$

$W = \int_{.2}^{.3} 50x\,dx \ J$

$W = \int_{.2}^{.3} 50x\,dx = 50 \cdot \frac{x^2}{2} \Big|_{.2}^{.3}$

$= 25x^2 \Big|_{.2}^{.3} = 25(.3)^2 - 25(.2)^2 = 1.25\,J$

Good

4. Set up an integral for the future value (supposing 6% continuous interest) after 20 years of an investment in an artist's music royalties that pays \$1,000,000 per year right now, but is expected to grow by 10% each year.

$FV = \int_0^n P(t)\, e^{r(n-t)}\, dt$

$\downarrow$

$FV = \int_0^{20} 1{,}000{,}000\,(1.1)^t\, e^{0.06(20-t)}\, dt$

Great

5. Find $\bar{x}$, the x-coordinate of the centroid of the first-quadrant portion of a circle with radius 1 centered at the origin.

$x^2+y^2=1$

$y=\sqrt{1-x^2}$

$$\bar{X} = \frac{1}{A}\int_a^b x[f(x)]\,dx = \frac{1}{\pi/4}\int_0^1 x\left(\sqrt{1-x^2}\right)dx$$

$U = 1-x^2$

$-\frac{du}{2x} = dx$

$$= \frac{4}{\pi}\int_0^1 x\left(u^{1/2}\right)\left(-\frac{du}{2x}\right) = \frac{-2}{\pi}\int_0^1 u^{1/2}\,du = \frac{-2}{\pi}\cdot\frac{2}{3}u^{3/2}\Big|_0^1$$

$$= -\frac{4}{3\pi}(0-1) = \boxed{\frac{4}{3\pi}}$$

Nice!

6. Use separation of variables to show that the general solution to the differential equation $\frac{dP}{dt} = kP$ is $P(t) = Ae^{kt}$.

$$\frac{dP}{dt} = kP$$

$$\int \frac{1}{P}\,dP = \int k\,dt$$

$$\ln|P| = k\cdot t + C$$

$$|P| = e^{kt+C}$$

$$|P| = e^{kt}\cdot e^{C}$$

$$P = A\cdot e^{kt}$$ (where A is either e^{C} or $-e^{C}$)

7. Bunny is a Calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, Calc is so impossible! It's totally unfair! I think I'm going to die! Our test had this question, like, where they gave us the integral but instead of working it out we were supposed to say, like, what it would answer, like other than just the area under it. How am I supposed to know what question $\int_{-2}^{2} 2\pi\sqrt{4-x^2}\sqrt{1+\left(\frac{-x}{\sqrt{4-x^2}}\right)^2}\,dx$ answers! It's cray-cray!"

Help Bunny out by explaining what question her integral might answer.

~~2π = volume?~~ $f(x) = \sqrt{4-x^2}$

~~$4-x^2$ = around y=4~~

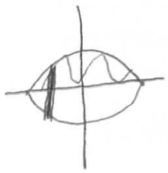
$2\pi\sqrt{\ }\ \sqrt{1+\ }$ = Surface area

By looking at the integral the first instinct is to say its used to find volume because of the 2π that is present. However after a closer look under the radicals it is easy to see that this integral is used to find the surface area of $f(x)$. This is known because $f(x)$, or $\sqrt{4-x^2}$, is not only being multiplied by 2π but also $\sqrt{1+f'(x)^2}$. The second radical is what makes this integral a surface area integral.

Good!

$y = \sqrt{1-x^2}$

9. Suppose that the top half of $x^2 + y^2 = 1$ is rotated around the x–axis. Set up an integral for the volume of the resulting solid and use it to find that volume.



discs $\to \int_a^b \pi (f(x))^2 \, dx$

symmetry $\to 2 \int_0^1 \pi \left(\sqrt{1-x^2}\right)^2 dx$

$2 \int_0^1 \pi \; 1-x^2 \, dx$

$\pi \left[x - \frac{x^3}{3} \right]_0^1$

$\pi \left[1 - \frac{(1)^3}{3} \right] - 0$

$2 \left[\frac{2}{3} \pi \right] = \frac{4}{3} \pi$

Good

10. (a) A large rectangular pool at a water resort is 30 meters by 20 meters and 3 meters deep. Write an integral for the amount of work required to pump all the water out over the edge.

$A_s = 600 \ m^2$

$V_s = 600 \Delta x \ m^3$

$m_s = 600000 \Delta x \ kg$

$F_s = 5{,}880{,}000{,}000 \ \Delta x \ N$

$W_s = F_s x$

$W_{tot} = \int_0^3 5{,}880{,}000{,}000 \ x \ dx$

Good

(b) A large rectangular wave pool at a water resort is 30 meters wide and 3 meters deep at the west end where the wave generators are located. For the first 20 meters from that west edge the depth is constant, but from there it slopes linearly up until 50 meters from the west edge the depth is 0. The drain malfunctions, so the entire pool needs to be drained by pumping the water out over the edge. Write an integral for the work required.

$A_s = 30(50-10x) \ m^2$

$V_s = 30(50-10x) \ \Delta x \ m^3$

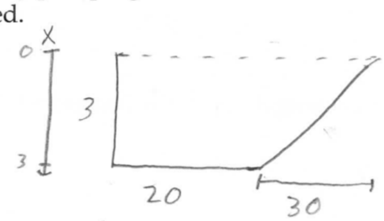
$m_s = 30000(50-10x) \ \Delta x \ kg$

$F_s = 294000 \ (50-10x) \ \Delta x \ N$

$W_s = F_s x$

$W_{tot} = \int_0^3 294000 \ (50-10x) x \ dx$

Excellent!



At $x=0$ $L=50$ At $x=3$ $L=20$

$L = 50-10x$