Exam 3 **Calculus 2** **4/1/22**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Determine the exact sum of the geometric series

$$5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \dots$$

$-2/3$ $-2/3$

$5 \cdot \frac{-2}{3} = \frac{-10}{3}$

$-\frac{10}{3} \cdot \frac{-2}{3} = \frac{+20}{9}$

$S = \frac{a}{1-r}$

If $|r| < 1$

Great!

$r = -2/3$ ✓

$|-2/3| < 1$

$1 + 2/3 = \frac{3}{3} + \frac{2}{3} = 5/3$

$S = \frac{5}{1-(-2/3)} = \frac{5}{1+2/3}$

$= 5/(5/3) = \frac{3}{\cancel{5}} \cdot \frac{\cancel{5}}{1} = 3$

$\boxed{S = 3}$

2. Find the first 3 partial sums of the series

$$\sum_{n=1}^{\infty} \frac{2}{(2n-1)!}$$

$S_1 = \frac{2}{(2(1)-1)!} = 2$

$S_2 = 2 + \frac{2}{(2(2)-1)!} = \frac{7}{3}$

$S_3 = \frac{7}{3} + \frac{2}{(2(3)-1)!} = \frac{47}{20}$

Great!

$1 + x + \frac{x^2}{2}$

3. Find the 5th degree MacLaurin polynomial for $f(x) = e^x$.

$f(x) = e^x \to f(0) = 1$

$f'(x) = e^x \to f'(0) = 1$

$f''(x) = e^x \to f''(0) = 1$

$f'''(x) = e^x \to f'''(0) = 1$

$f^4(x) = e^x \to f^4(0) = 1$

$f^5(x) = e^x \to f^5(0) = 1$

$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 \ldots$

$\downarrow$

$$1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!}$$

Great

4. Determine whether the series $\sum_{n=2}^{\infty} \frac{3}{n \ln n}$ converges or diverges.

Integral test

$$\lim_{b \to \infty} \int_2^b \frac{1}{x \ln x} = \lim_{b \to \infty} \int_2^b \frac{1}{x \cdot u} x\,du = \lim_{b \to \infty} \int_2^b \frac{1}{u}\,du = \lim_{b \to \infty} \ln|u| \Big|_2^b$$

u sub

$\ln x = u$

$\frac{1}{x}dx = du$

$dx = x\,du$

$$\lim_{b \to \infty} \ln|\ln(x)| \Big|_2^b = \lim_{b \to \infty} \left[\ln|\ln(2)| - \ln|\ln(b)| \right] = \text{diverges}$$

$\downarrow \infty$

$\sum_{n=2}^{\infty} \frac{3}{n \ln n}$ diverges by the integral test

Excellent!

5. Determine whether the series $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{\sqrt{n}}$ converges or diverges.

1 or 3, no AST

$\sqrt{n} \to n^{\frac{1}{2}}$

is similar to $\frac{1}{n^{\frac{1}{2}}}$ which diverges based on the p-series as $p = \frac{1}{2} < 1$

$\therefore$ because $\frac{1}{n^{\frac{1}{2}}} \leq \frac{2+(-1)^n}{\sqrt{n}}$, and $\frac{1}{n^{\frac{1}{2}}}$ diverges,

$\sum_{n=1}^{\infty} \frac{2+(-1)^n}{\sqrt{n}}$ diverges by the comparison test

Excellent!

6. Write the first 4 non-zero terms of the Taylor Series for $f(x) = \sin x$ centered at $x = \frac{\pi}{2}$.

$f(x) = \sin x$ $\quad f(\pi/2) = 1$

$f''(x) = -\sin x$ $\quad f''(\pi/2) = -1$

$f^{(4)}(x) = \sin x$ $\quad f^{(4)}(\pi/2) = 1$

$f^{(6)}(x) = -\sin x$ $\quad f^{(6)}(\pi/2) = -1$

$T_4 = 1 - \frac{(x-\frac{\pi}{2})^2}{2!} + \frac{(x-\frac{\pi}{2})^4}{4!} - \frac{(x-\frac{\pi}{2})^6}{6!}$

Excellent!

7. Biff is a Calculus student at Enormous State University, and he's having some trouble. Biff says "Dude, this series stuff is crazy. So we were supposed to say if $\sum_{n=2}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges or diverges, right? So I said it diverges by comparison to $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$, but this other person said it converges by geometric. So I figure that makes it a tie, right?"

Help Biff out by explaining the validity of his two different justifications. You do not need to say whether the series converges or not; you're just commenting on the proposed justifications.

First Biff's comparison method is invalid because all terms must be positive for comparison test to be valid and $\sum_{n=2}^{\infty} = \frac{1}{\sqrt{2}} + \underbrace{\frac{-1}{\sqrt{3}}}_{\text{not positive}}$ has negative terms.

and for it to converge by geometric the values need to be changing by a constant rate

$\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{2}$ are the first 3 terms

$$\frac{\frac{1}{\sqrt{2}} X}{\frac{1}{\sqrt{2}}} = \frac{-\frac{1}{\sqrt{3}}}{\frac{1}{\sqrt{2}}}$$

$$\underbrace{X = \sqrt{2} \cdot -\frac{1}{\sqrt{3}}}_{\text{if this is geometric this would be the } r}$$

$-\frac{1}{\sqrt{3}} \cdot \left(\sqrt{2} \cdot -\frac{1}{\sqrt{3}}\right) \approx .4714 \neq \frac{1}{2}$

so it is not increasing by a constant rate meaning this is not a geometric series.

Good

So both tests for convergence are invalid

8. Use a Maclaurin polynomial with at least 4 terms to approximate $\sqrt[3]{e}$.

Know $e^x \approx 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

$e^{1/3}$, plug $1/3$ in for x

$e^{1/3} \approx 1 + \frac{1}{3} + \frac{(1/3)^2}{2!} + \frac{(1/3)^3}{3!} = 1.39506$

Excellent!

9. Does $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)3^n}$ converge when $x = 1$?

Using the Rat. Test...

$$\lim_{n\to\infty} \left| \frac{\frac{1^{n+1}}{(n+2)3^{n+1}}}{\frac{1^n}{(n+1)3^n}} \right| = \lim_{n\to\infty} \left| \frac{(n+1)}{(n+2)\cdot 3} \right|$$

$$\stackrel{L'h}{=} \left|\frac{1}{3}\right| \lim_{n\to\infty} |1| = \left|\frac{1}{3}\right| < 1$$

$\therefore$ The series converges when $x = 1$

Excellent!

10. Find the interval of convergence of $\sum_{n=0}^{\infty} \frac{x^n}{(n+1)3^n}$.

Using the Rat. Test...

$$\lim_{n\to\infty} \left| \frac{\frac{x^{n+1}}{(n+2)3^{n+1}}}{\frac{x^n}{(n+1)3^n}} \right| = \lim_{n\to\infty} \left| \frac{x(n+1)}{3(n+2)} \right| \overset{L'h}{=} \left|\frac{x}{3}\right| < 1$$

$$|x| < 3$$

1st Draft $(-3, 3)$

Rat. Test w/ $x = -3$

$$\lim_{n\to\infty} \left| \frac{\frac{(-3)^{n+1}}{(n+2)3^{n+1}}}{\frac{(-3)^n}{(n+1)3^n}} \right| = \lim_{n\to\infty} \left| \frac{3(n+1)}{3(n+2)} \right| = 1 \text{ (Inconclusive)}$$ (This never helps!)

Using the A.S.T. with $x = -3$

- Alternating signs ✓
- $\lim_{n\to\infty} \frac{3^n}{(n+1)3^n} = 0$ (sequence tends to 0) ✓
- $\left(\frac{3^n}{(n+1)3^n}\right)' = [(n+1)^{-1}]' = -(n+1)^{-2}$ (sequence is decreasing) ✓

$\therefore$ The series converges when $x = -3$

For $x = 3$, using the integral test,-

$$\int_0^{\infty} \frac{3^n}{(n+1)3^n}\,dn = \lim_{b\to\infty} \int_0^b \frac{1}{n+1}\,dn = \lim_{b\to\infty} \ln|b+1| - \ln|1| = \infty$$

The integral diverges, so the series also diverges when $x = 3$

The interval of convergence is $[-3, 3)$

Excellent!