Examlet 2 **Foundations of Advanced Math** **2/25/22**

1. Mark each of the following statements as true or false:

   (a) $(5,9) \cap (8,10) = (8,9)$ — **T** F

   $(8,9)$

   (b) $\{5,9\} \cap \{8,10\} = \{8,9\}$ — T **F**

   $\varnothing$

   (c) $[5,9] \cup [8,10] = [8,9]$ — T **F**

   $[5,10]$

   (d) $(5,9) - (8,10) = (5,8)$ $(5,8]$ — T **F**

   $(5,8]$

   (e) $\{5,9\} - \{8,10\} = \{5,7\}$ — T **F**

   $\{5,9\}$

   (f) $\varnothing \in \{1,2,3\}$ — T **F**

   $\varnothing \in P(\{1,2,3\})$

   (g) $\varnothing \subseteq \{1,2,3\}$ — **T** F

   (h) $\{2\} \in \{1,2,3\}$ — T **F**

   $2 \in \{1,2,3\}$

   $\{2\} \in P(\{1,2,3\})$

   (i) $\{2\} \subseteq \{1,2,3\}$ — **T** F

   (j) $\mathcal{P}(\{a,b\}) = \{\{\varnothing\}, \{a\}, \{b\}, \{a,b\}\}$ — T **F**

or true if

Since $\varnothing = \{\}$

you meant

$\{\varnothing, \{a\}, \{b\}, (a,b)\}$

Since $\{\varnothing\}$ doesn't make much sense

2. For each $x \in \mathbb{R}^+$, let $W_x = \left\{ y \,\middle|\, \frac{1}{x} \le y < \frac{3x+1}{x} \right\}$. What are:

(a) $\bigcup_{x \in \{1,2,3\}} W_x$

$\left[\frac{1}{3}, 4\right)$

(b) $\bigcap_{x \in \{1,2,3\}} W_x$

$\left[1, \frac{10}{3}\right)$

(c) $\bigcup_{x \in \mathbb{Z}^+} W_x$

$(0, 4)$

Great

(d) $\bigcap_{x \in \mathbb{Z}^+} W_x$

$[1, 3]$

(e) $\bigcup_{x \in \mathbb{R}^+} W_x$

$\mathbb{R}^+$

(f) $\bigcap_{x \in \mathbb{R}^+} W_x$

$1 \le y < 4$

0.1

$\emptyset$

$10 \le y \le 10 + 3$

3. $(A \cup B)' = A' \cap B'$

Proof:

Take $x \in (A \cup B)'$ where means that

$\neg(x \in (A \cup B))$. where is same as

$\neg(x \in A \lor x \in B)$ and it is logically equals to

$\neg(x \in A) \land \neg(x \in B)$

$\Downarrow$

$x \in A' \land x \in B'$ which can be written as

$x \in A' \cap B'$ $\therefore$ $(A \cup B)' \subseteq A' \cap B'$.

Now take $x \in A' \cap B'$ which means

$x \in A' \land x \in B' \Rightarrow$ $\neg(x \in A) \land \neg(x \in B)$

and it is logically equals to $\neg(x \in A \lor x \in B)$.

whichis same as $\neg(x \in (A \cup B)) \Rightarrow x \notin (A \cup B) \Rightarrow x \in (A \cup B)'$

$\therefore$ $A' \cap B' \subseteq (A \cup B)'$

therefore $(A \cup B)' = A' \cap B'$ Great

4. Show that if $b \in \mathbb{R}$ with $b > 0$, then $\forall n \in \mathbb{N}, b^n > 0$.

Using induction

Base Case: $n = 0$ $\quad b^0 = 1 \quad 1 > 0$ Base case is true

Inductive hypothesis: $b^n > 0$, for $n \in \mathbb{N}$

Inductive step: $b^n > 0$ using inductive hypothesis

$b^n b > (0)(b)$ comparison multiplication principle

$b^{n+1} > 0$

Since the statement is true for the base case, and if it is true for n then it is true for $n+1$, then the statement is true for all $n \in \mathbb{N}$.

Lemma 2

5. $\forall x, y \in \mathbb{R}, |x| < y \Rightarrow -y < x < y$.

By definition, $|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$, so we have 2 cases.

1) $x \geq 0$. So $|x| = x$, which means $x < y$. Since we also know $x \geq 0$, we can say $0 \leq x < y \leftrightarrow 0 < y$. By CAP we can get $0 - y < y - y \leftrightarrow -y < 0$. Now, we have $-y < 0$ and $x \geq 0$, so by transitive property we know $-y < x$. We can combine this with $x < y$ to get $-y < x < y$.

2) $x < 0$. So $|x| = -x$, which means $-x < y$. We know $x < 0$, so by CAP we get $0 < -x$. So by the transitive property, $0 < y$. By subtracting y from both sides by CAP we have $-y < 0$. By the transitive property with $0 < -x$ and $-y < 0$ we know $-y < -x$. So, we can add x and y to both sides by CAP to get $x + y - y < x + y - x \leftrightarrow x < y$. Since we know $-x < y$, we can add x and $-y$ to both sides by CAP to get $-x + x - y < y + x - y \longleftrightarrow -y < x$. Now, combining the two gives us $-y < x < y$.

Since it is true for both cases and the cases are exhaustive, the statement is true. $\square$

Nice!