Examlet 4 **Foundations of Advanced Math** **4/15/22**

1. (a) State the definition of a reflexive relation.

a relation "$\sim$" is reflexive iff $a \sim a$. $\forall a \in A$.
A being a set.

Good

(b) Give an example of a relation on the set $\{a, b, c\}$ which is reflexive but not symmetric and not transitive.

$R = \{(a,a), (b,b), (c,c), (a,b), (b,a), (b,c)\}$

Yep.

2. (a) Suppose that $\sim$ is an equivalence relation on the set $A = \{a, b, c, d, e\}$ and that $[a] = \{a, b, c\}$ and $[d] = \{d, e\}$. Write the partition Π corresponding to $\sim$.

$\Pi = \{\{a,b,c\}, \{d,e\}\}$

Great

or

$\Pi = \{[a], [d]\}$

(b) Suppose that Π is the partition $\{\{1\}, \{2, 4\}, \{3, 5\}\}$ of the set $A = \{1, 2, 3, 4, 5\}$. Find the relation $\sim$ corresponding to Π, expressing it as a set of ordered pairs.

$\{(1,1), (2,2), (2,4), (4,2), (4,4), (3,3), (5,5), (3,5), (5,3)\}$

Good

3. Let S be a set and Π a partition of S. Let $\sim$ be a relation on S defined by $a \sim b \Leftrightarrow \exists P \in \Pi$ for which $a, b \in P$.

(a) Show $\sim$ is a reflexive relation.

Take $a \in S$. Π is a partition of S, so the union of all the sets in Π is all of S. Then $\exists P \in \Pi$ such that $a \in P$ (or $a, a \in P$), so $a \sim a$. Therefore the relation is reflexive. □

(b) Show $\sim$ is a symmetric relation.

Suppose we have that $a \sim b$, which means that $\exists P \in \Pi$ such that $a, b \in P$. Since $a, b \in P$, we can also say that $b, a \in P$. Therefore $b \sim a$ so the relation is symmetric. □

(c) Show $\sim$ is a transitive relation.

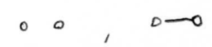
Suppose we have that $a \sim b$ and $b \sim c$, meaning that $\exists P \in \Pi$ such that $a, b \in P$ and $\exists Q \in \Pi$ such that $b, c \in Q$. A partition consists of pairwise disjoint non-empty sets, so because we have that $b \in P$ and $b \in Q$, then it must be that $P = Q$. Since $a, b \in P$ and $b, c \in Q$, then $a, b, c \in P \wedge a, b, c \in Q$. Therefore $a \sim c$ so the relation is transitive. □

Very Nice!

4. (a) Give all (unlabeled) graphs with $n \le 4$ vertices.

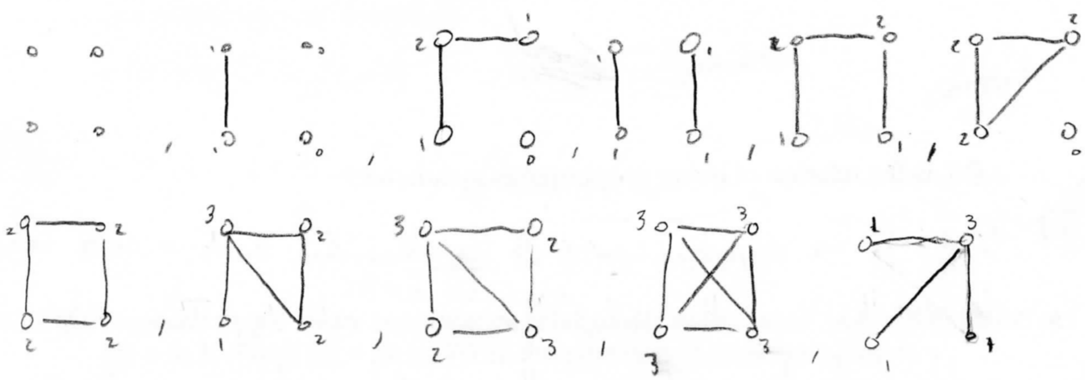
$n=0$:

$n=1$:

$n=2$: 

$n=3$: 

Excellent!

$n=4$: 

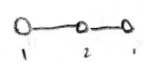
(b) Give all (unlabeled) trees with $n \le 4$ vertices.

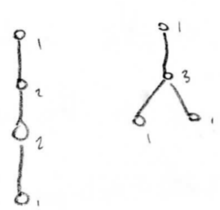
$n=0$:

$n=1$:

$n=2$: 

Great

$n=3$: 

$n=4$: 

5. Say that two vertices v_1 and v_2 of a graph G are **propinquous** iff there exists a walk between them that contains exactly one vertex other than v_1 and v_2.

(a) Is the relation of being propinquous reflexive?

Take the single vertex v_1. By definition, this single vertex can be considered a walk to itself. However, there is not another vertex present in the walk, so being propinquous is not a reflexive relation. □

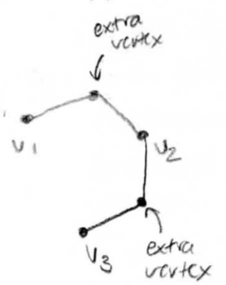
Good

(b) Is the relation of being propinquous symmetric?

Suppose we have that v_1 and v_2 are propinquous, so there exists a walk from V_1 to V_2 with another vertex in the walk. Since there is a walk from V_1 to V_2, then there must also be a walk from V_2 to V_1, including that same extra vertex. The path from V_2 to V_1 will simply be the path from V_1 to V_2 in reverse order. Therefore the relation is symmetric. □

Good

(c) Is the relation of being propinquous transitive?



In this graph, there exists a walk from V_1 to V_2 that contains exactly one other vertex and there is a walk from V_2 to V_3 with exactly one other vertex. This means that V_1 and V_2 are propinguous and V_2 and V_3 are propinguous. However, a walk from V_1 to V_3 would contain at least three other vertices. Thus V_1 and V_3 are not propinquous so the relation is not transitive. □

Good