Exam 2 **Calc 1** **3/3/23**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. State the formal definition of the derivative of a function $f(x)$.

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

Good

2. Use the following table of values for $f(x)$ and $g(x)$ to find values for the following:

x	1	2	3	4	5	6
$f(x)$	6	3	5	1	4	2
$g(x)$	2	1	6	3	5	4
$f'(x)$	2	4	1	5	7	8
$g'(x)$	5	9	7	11	2	12

(a) If $h(x) = f(x) \cdot g(x)$, what is $h'(2)$ and why?

Product rule $= f' \cdot g + g' \cdot f$

$h'(2) = f'(2) \cdot g(2) + g'(2) \cdot f(2)$

$4 \cdot 1 + 9 \cdot 3$

$4 + 27 = \boxed{31}$

Good

(b) If $h(x) = \frac{f(x)}{g(x)}$, what is $h'(5)$ and why?

Quotient rule $= \frac{f' \cdot g - g' \cdot f}{g^2}$

$$h'(5) = \frac{f'(5) \cdot g(5) - g'(5) \cdot f(5)}{g(5)^2} = \frac{7 \cdot 5 - 2 \cdot 4}{5^2} = \frac{35 - 8}{25} = \boxed{\frac{27}{25}}$$

(c) If $h(x) = f(g(x))$, what is $h'(3)$ and why?

$h'(3) = f'(g(3)) \cdot g'(3)$

$= f'(6) \cdot g'(3)$

$8 \cdot 7 = \boxed{56}$

3. Find the derivatives of the following functions:

(a) $f(x) = x^2 \sin x$ product

$(2x \cdot \sin x) + (x^2 \cdot \cos x)$

$= 2x \sin x + x^2 \cdot \cos x$

$f'(x) = 2x \sin x + x^2 \cos x$

(b) $g(x) = \frac{\sin x}{x^2}$ quotient rule

$\frac{(\cos x \cdot x^2) - (\sin x \cdot 2x)}{(x^2)^2} = \frac{x^2 \cos x - 2x \sin x}{(x^2)^2}$

Great

W

$\cos x \leftarrow$ out, in $\rightarrow 2x$

(c) $h(x) = \sin(x^2)$ Chain rule

$h'(x) = \cos(x^2) \cdot 2x$

$= 2x\cos(x^2)$

$\frac{\sin}{\cos} \rightarrow \frac{1}{\cos}$

4. Show why the derivative of $\tan x$ is $\sec^2 x$.

$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)'$ * quotient rule

$= \frac{(\cos x \cdot \cos x) - (\sin x \cdot -\sin x)}{\cos^2 x}$

W

$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$ } using $\cos x + \sin x = 1$ we know this top expression is 1

$= \frac{1}{\cos^2 x}$ or $\sec^2 x$

using the trig identity

Good

5. Use the definition of the derivative to find the derivative of $f(x) = \dfrac{1}{x}$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{x+h} - \frac{1}{x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{x}{(x+h)x} - \frac{x+h}{(x+h)x}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{-h}{(x+h)x}}{h}$$

Excellent!

$$= \lim_{h \to 0} \frac{-1}{(x+h)x}$$

$$f'(x) = \frac{-1}{x^2}$$

So the derivative of $f(x) = \frac{1}{x}$ is $f'(x) = \frac{-1}{x^2}$

6. State and prove the Product Rule.

Product rule: $(f \cdot g)' = f'g + g'f$

Let $F(x) = f(x) \cdot g(x)$

$$F'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x) \cdot g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) + f(x+h) g(x) - f(x+h) g(x) - f(x) \cdot g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) \cdot g(x+h) - f(x+h) \cdot g(x) + f(x+h) \cdot g(x) - f(x) \cdot g(x)}{h}$$

$$= \lim_{h \to 0} \left\{ \frac{f(x+h) \cdot \left[g(x+h) - g(x)\right] + g(x)\left[f(x+h) - f(x)\right]}{h} \right\}$$

$$= \lim_{h \to 0} \left[f(x+h) \cdot \frac{g(x+h) - g(x)}{h} + g(x) \frac{f(x+h) - f(x)}{h} \right]$$

$$= \lim_{h \to 0} f(x+h) \cdot \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} + \lim_{h \to 0} g(x) \cdot \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f'g + g'f \;!$$

yeh!

Well done!

7. Bunny is a calculus student at Enormous State University, and she's having some trouble. Bunny says "OMG! Why does calculus have to be so confusing, like, they're literally trying to kill us? The professor definitely said that the derivative of 5 is zero on Monday, but then totally said that the derivative of 5 is 5 on Wednesday. Is it really different on different days?"

Help Bunny by explaining as clearly as you can what's going on.

When looking for derivative you are looking for a rate. The derivative of 5 is 0 because 5 is a constant, there is no rate and 5 doesn't change. The derivative of 5x is 5 which most likely was where the confusion lies. 5x has a rate of increase of 5 which is the derivative.

Good

8. Use a local linearization for the function $f(x) = x^{2/3}$ to approximate $(8.1)^{2/3}$.

$f'(x) = \frac{2}{3} x^{-1/3}$ $\qquad$ $f(8) = 8^{2/3} = 2^2 = 4$

$f'(8) = \frac{2}{3} \cdot 8^{-1/3} = \frac{2}{3} \cdot \frac{1}{2} = \frac{1}{3}$

Tangent Line:

$$y - 4 = \frac{1}{3}(x - 8)$$

$$y = \frac{1}{3}(x - 8) + 4$$

Linearization:

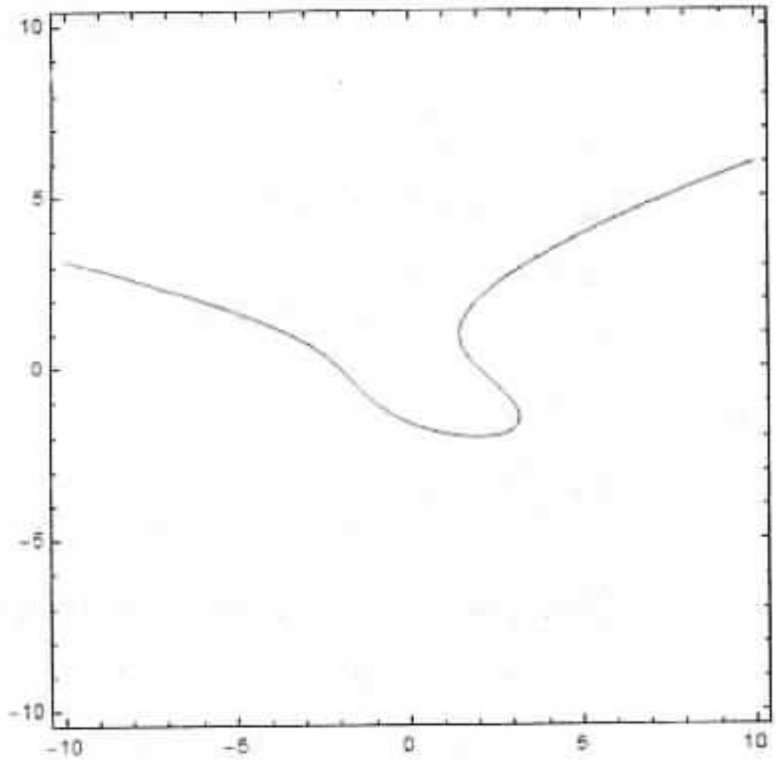
$$L(x) = \frac{1}{3}(x - 8) + 4$$

So $(8.1)^{2/3} \approx \frac{1}{3}(8.1 - 8) + 4$

$$= \frac{1}{3} \cdot 0.1 + 4$$

$$= 4.0\overline{3}$$

9. Find an equation of the line tangent to $2xy + x^2 - y^3 = 4$ at the point $(3, -1)$.



$$2 \cdot y + 2x \cdot y' + 2x - 3y^2 \cdot y' = 0$$

$$2x \cdot y' - 3y^2 \cdot y' = -2y - 2x$$

$$y'(2x - 3y^2) = -2y - 2x$$

$$y' = \frac{-2y - 2x}{2x - 3y^2}$$

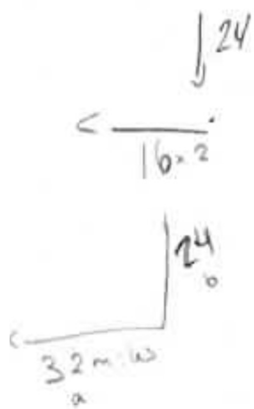
So at $(3,-1)$, $y' = \dfrac{-2(-1) - 2(3)}{2(3) - 3(-1)^2} = \dfrac{2 - 6}{6 - 3} = \dfrac{-4}{3}$

Tangent Line:

$$y - (-1) = \frac{-4}{3}(x - 3)$$

10. A train leaves Boston heading west at noon travelling 16 mi/hr. A second train is heading toward Boston from the north at 24 mi/hr and will arrive at 3pm. How fast is the distance between the trains changing at 2pm?

$$32^2 + 24^2 = 40^2$$
$$1024 + 576 = 1600$$



$$2(32)\cdot(16) + 2(24)\cdot(-24) = 2(40)\frac{dc}{dt}$$
$$64\cdot 16 + 48\cdot -24$$
$$1024 - 1152 = 2(40)\frac{dc}{dt}$$
$$\frac{-128}{80} = \frac{80}{80}\frac{dc}{dt}$$
$$-1.6 = \frac{dc}{dt}$$

Well done!