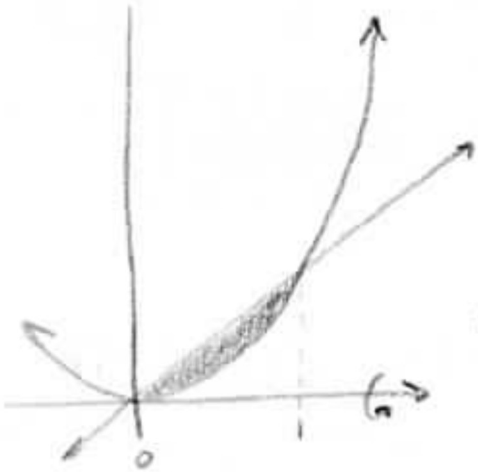
Exam 2 **Calculus 2** **3/1/23**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Set up an integral for the area of the region bounded between $y = x^2$ and $y = x$.



$$\text{Area} = \int_0^1 x - x^2 \, dx$$

Nice!

2. Set up an integral for the volume of the solid obtained when the region from #1 is rotated around the x-axis.

$$\text{Volume} = \pi \int_0^1 (x)^2 - (x^2)^2 \, dx$$

Good

3. A force of 5 pounds is required to hold a spring stretched 0.6 feet beyond its natural length. How much work (in foot-pounds) is done in stretching the spring from its natural length to 0.7 feet beyond its natural length?

$f = k \cdot x \quad W = f \cdot d$

$\frac{5}{.6} = \frac{k \cdot .6}{.6}$

$k = 8.3$

$\int_0^{.7} 8.3x \, dx$

$\left. \frac{8.3x^2}{2} \right|_0^{.7}$

$\frac{8.3(.7)^2}{2} = \frac{4.067}{2} = 2.0335$ ftlbs

Great

4. Set up an integral for the future value (supposing 5% continuous interest) after 15 years of investing $10,000 per year.

$F.V. = \int_0^m P(t) e^{r(m-t)}$

Great

$$\boxed{\int_0^{15} 10{,}000 \, e^{0.05(15-t)} \, dt}$$

5. Set up an integral for the length of the curve $y = \cos x$ from one peak to the next.

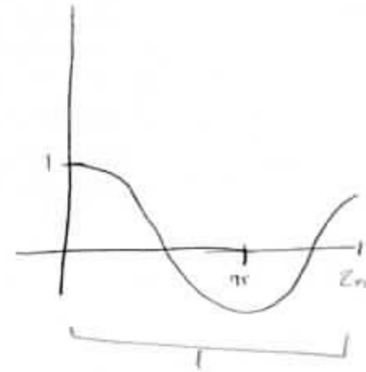
$$\int_0^{2\pi} \sqrt{1+(-\sin(x))^2}\,dx$$

$\int_a^b \sqrt{1+(f'(x))^2}\,dx$

$f(x) = \cos x$

$f'(x) = -\sin x$

Great



assuming you mean here

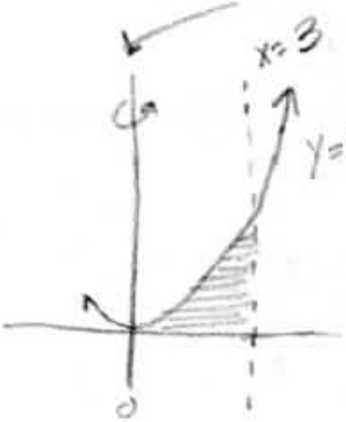
6. Set up integrals for the x coordinate of the center of mass of the region bounded between $y = x^2$ and $y = x$ from #1.

$$\text{center of mass} = \frac{\int_0^1 x\cdot(x-x^2)\,dx}{\int_0^1 x-x^2\,dx} = \frac{1}{2}$$

7. Bunny is a Calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod, Calc is so impossible! It's totally unfair and I'm going to die! The professor in class today was saying that we know these two integrals have to give the same answer because they're the same thing, but they look totally different to me. She was saying that $\int_0^3 2\pi(x)(x^2)dx$ would have to give the same answer as $\int_0^9 \left[\pi(3)^2 - \pi(\sqrt{y})^2\right] dy$, but I think they look totally different!"

Help Bunny out by explaining why we might be able to tell (other than working them out) that her two integrals have the same value.

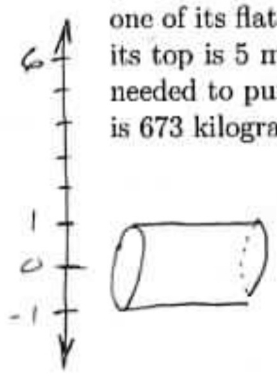
First lets think about the graphs of $x=3$ and $y=x^2$, they look something like this. we know by looking at the given equations that we are trying to do solids by revolution since they match our formulas (shells $= 2\pi\int_a^b (\text{height})(\text{radius})dx$ and washers $= \pi\int_0^b (\text{outer})^2 - (\text{inner})^2\,dy$). Thinking about the solid formed we know it can be done with either shells



or washers, but if we do it with washers we have to use a dy integral which can be tricky. Using a shells integral we know the radius will be x and the height will be x^2, so we can simply plug those in to ges $\int_0^3 2\pi(x)(x^2)\,dx$. solving both integrals we see we will get the same answer.

Great

8. A gas station stores its gasoline in a tank under the ground. The tank is a cylinder lying horizontally on its side. (In other words, the tank is not standing vertically on one of its flat ends.) If the radius of the cylinder is 1 meter, its length is 7 meters, and its top is 5 meters under the ground, set up an integral for the total amount of work needed to pump the gasoline out of the tank to ground level. (The density of gasoline is 673 kilograms per cubic meter; use $g = 9.8m/s^2$).



Width of a slice $= 2 \cdot \sqrt{1-x^2}$ m

Area " " " " $= 2\sqrt{1-x^2} \cdot 7\, m^2$

Volume " " " " $= 14\sqrt{1-x^2}\, \Delta x\, m^3$

Mass " " " " $= 14\sqrt{1-x^2} \cdot 673\, \frac{kg}{m^3} \cdot m^3\, \Delta x$

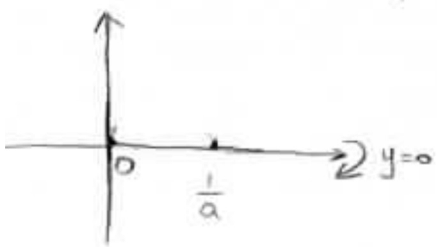
Force " " " $= 9422\sqrt{1-x^2}\, kg \cdot 9.8\, m/s^2\, \Delta x$

Work " " " $= 92335.6\sqrt{1-x^2} \cdot (6-x)\, \Delta x$

$$\text{Total Work} = \int_{-1}^{1} 92335.6\sqrt{1-x^2}\,(6-x)\,dx$$

$\int_a^b \pi[f(x)]^2\,dx$

9. Let $a > 0$. Show that the volume obtained when the region between $y = a\sqrt{x - ax^2}$ and $y = 0$ is rotated around the x-axis is independent of the constant a.



Since $a > 0$
when $x < 0$,
$x - ax^2 < 0$, cannot get that.

$$\int_0^{\frac{1}{a}} \pi \left(a\sqrt{x-ax^2}\right)^2 dx$$

$x - ax^2 = 0$
$1 - ax = 0$
$ax = 1$
$x = \frac{1}{a} \mid 0$

$$= \pi \int_0^{\frac{1}{a}} a^2(x - ax^2)\,dx$$

$$= a^2\pi \int_0^{\frac{1}{a}} x - ax^2\,dx$$

$$= a^2\pi \left(\frac{1}{2}x^2 - \frac{a}{3}x^3\right)\Big|_0^{\frac{1}{a}}$$

$$= a^2\pi \left(\frac{1}{2} \times \frac{1}{a^2} - \frac{a}{3}\cdot\frac{1}{a^3}\right)$$

$$= a^2\pi \times \left(\frac{1}{2a^2} - \frac{1}{3a^2}\right)$$

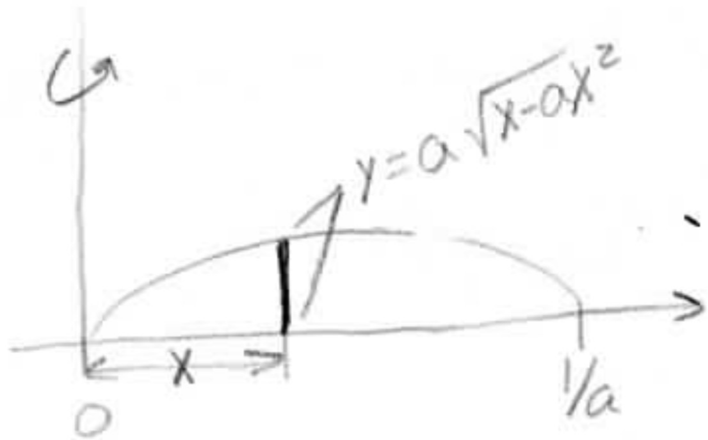
Excellent!

$$= a^2\pi \times \frac{1}{2a^2} - a^2\pi \times \frac{1}{3a^2}$$

$$= \frac{\pi}{2} - \frac{\pi}{3}$$

$$= \frac{\pi}{6}$$

10. Let $a > 0$. Write an integral for the volume of the solid obtained by rotating the region between $y = a\sqrt{x - ax^2}$ and $y = 0$ around the y-axis.



$$\text{Volume} = 2\pi \int_0^{1/a} x\left(a\sqrt{x - ax^2}\right)dx$$

Great!