Exam 3 **Calculus 2** **3/31/23**

Each problem is worth 10 points. For full credit give good justification for your answers.

1. Determine the exact sum of the geometric series

$$S = \frac{a}{1-r} \text{ If } |r| < 1 \qquad 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \ldots \qquad a = 1, \; r = -1/2$$

$$S = \frac{1}{1-(-1/2)} = \frac{1}{3/2} = \frac{1}{1} \cdot \frac{2}{3} = \boxed{\frac{2}{3}}$$

Good

2. Find the first 3 partial sums of the series

$$\sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)}$$

$$S_0 = (-1)^0 \frac{1}{1} = 1$$

$$S_1 = (-1)^1 \frac{1}{2 \times 1 + 1} + 1 = -\frac{1}{3} + 1 = \frac{2}{3}$$

$$S_2 = (-1)^2 \frac{1}{2 \times 2 + 1} + \frac{2}{3} = \frac{1}{5} + \frac{2}{3} = \frac{3}{15} + \frac{10}{15} = \frac{13}{15}$$

$$S_3 = (-1)^3 \frac{1}{2 \times 3 + 1} + \frac{13}{15} = -\frac{1}{7} + \frac{13}{15}$$

$$= -\frac{15}{105} + \frac{91}{105}$$

$$= \frac{76}{105}$$

Excellent

3. Write the 5[th] degree MacLaurin polynomial for $f(x) = \sin x$.

$f(x) = \sin x \qquad f(0) = 0$

$f'(x) = \cos x \qquad f'(0) = 1$

$f''(x) = -\sin x \qquad f''(0) = 0$

$f^{(3)}(x) = -\cos x \qquad f^{(3)}(0) = -1$

$f^{(4)}(x) = \sin x \qquad f^{(4)}(0) = 0$

$f^{n}(0)\frac{x^n}{n!} \qquad f^{(5)}(x) = \cos x \qquad f^{(5)}(0) = 1$

$$0 + 1 \times \frac{x^1}{1!} + 0 - 1 \times \frac{x^3}{3!} + 0 + 1 \times \frac{x^5}{5!}$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!}$$

Nice!

4. Determine whether the series $\sum_{n=1}^{\infty} \frac{2+(-1)^n}{\sqrt{n}}$ converges or diverges.

Well, first off, I know $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ diverges because it's a p-series with $p \leq 1$.

So then notice $2 + (-1)^n$ is either 1 or 3, so it's always greater than or equal to 1.

So $\quad 2 + (-1)^n \geq 1$

$$\Rightarrow \quad \frac{2+(-1)^n}{\sqrt{n}} \geq \frac{1}{\sqrt{n}}$$

Then by the Comparison Test, this series consists of terms bigger than those of a series that diverges, so this must diverge also.

5. Determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n^3-1}$ converges or diverges.

$$\lim_{n\to\infty} \frac{\frac{1}{n^3-1}}{\frac{1}{n^3}}$$

$$= \lim_{n\to\infty} \frac{n^3}{n^3-1}$$

$$\underline{\underline{L'H}} \lim_{n\to\infty} \frac{3n^2}{3n^2}$$

$$= 1 > 0$$

Since $\sum_{n=2}^{\infty} \frac{1}{n^3}$, $\underline{p>1}$

So $\sum_{n=2}^{\infty} \frac{1}{n^3}$ converges

So $\sum_{n=2}^{\infty} \frac{1}{n^3-1}$ converges too

Good

6. Write the 4th degree Taylor polynomial for $f(x) = \ln x$ centered at $x = 1$.

$f(x) = \ln x$

$f'(x) = \frac{1}{x}$

$f''(x) = -\frac{1}{x^2} = (-x^{-2})$

$f'''(x) = \frac{2}{x^3} = (2x^{-3})$

$f''''(x) = \frac{-6}{x^4}$

$f(1) = 0$

$f'(1) = 1$

$f''(x) = -1$

$f'''(x) = 2$

$f^{(4)}(x) = -6$

$$\ln \approx (x-1) - \frac{(x-1)^2}{2} + \frac{\cancel{2}(x-1)^3}{\cancel{6}_3} - \frac{6(x-1)^4}{24_4}$$

Excellent!

$$\boxed{\ln \approx (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}}$$

4

$$\sum_{n=0}^{\infty} f^{(n)}(1) \frac{(x-1)^n}{n!}$$

7. Bunny is a Calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygawd! This series stuff is literally turning my brain into a donut! So, like, they had us find a Taylor thingy for ln, right? And it was centered at 1, right? So then they asked us to use it for $\ln 2$ and $\ln 4$, right? And the answer for $\ln 4$ was really far off, right? And they asked us, like, how high the degree needed to be to make it better, right? And I just cried."

Help Bunny out by explaining what happens when using higher and higher degree Taylor polynomials like this.

So Bunny, the thing you need to consider is the interval of convergence. Most times you're right that higher degree Maclaurin or Taylor polynomials do a better job. But if you use them outside their interval of convergence it's like using dairy products after their expiration dates - it could go really bad.

So you said your polynomial was centered at 1. Then 2 isn't too far away, but 4 is a lot farther. It turns out the interval of convergence for that series is $(0,2]$, so 4 is right out. That means higher degree polynomials can actually backfire and get worse and worse approximations.

8. Find the radius of convergence of $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$.

Rat. Test!

$$\lim_{n\to\infty}\left|\frac{(-1)^n\cdot\frac{x^{2(n+1)+1}}{2(n+1)+1}}{(-1)^n\cdot\frac{x^{2n+1}}{2n+1}}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{2n+3}\cdot\frac{2n+1}{x^{2n+1}}\right|$$

$$= \lim_{n\to\infty}\left|\frac{\cancel{x^{2n+1}}\cdot x^2}{2n+3}\cdot\frac{2n+1}{\cancel{x^{2n+1}}}\right| = |x^2|\cdot\lim_{n\to\infty}\left|\frac{2n+1}{2n+3}\right|$$

$$\overset{L'H}{=} |x^2|\cdot\lim_{n\to\infty}\left|\frac{2}{2}\right| = |x^2|$$

So this converges if $|x^2|<1$

$\boxed{\therefore \text{ the convergence radius is } 1}$

Excellent!

9. Find the interval of convergence of $\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$.

Test $x=1$

$$\sum_{n=0}^{\infty}(-1)^n\frac{(1)^{2n+1}}{2n+1}$$ Alternating Series test!

☑ Alternating signs

☑ decreasing

☑ $\lim_{n\to\infty}=0$

converges when $x=1$!

$(-1)^n$ means signs alternate!

$f(x)=(2n+1)^{-1}$ $f'(x)=-(2n+1)^{-2}\cdot 2$ ← negative derivative means decreasing!

$\lim_{n\to\infty}\frac{1}{2n+1}=\frac{1}{\infty}=0$

Test $x=-1$

-1 raised to an odd number will always be odd

$$\sum_{n=0}^{\infty}(-1)^n\frac{(-1)^{2n+1}}{2n+1}=\sum_{n=0}^{\infty}(-1)^n\cdot\frac{-1}{2n+1}$$ Alternating Series test!

By the same reasons we got from Plugging in 1 we know it also converges when $x=-1$

$\boxed{\therefore \text{ the interval of convergence is } [-1,1]}$

Great

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10. Use a Maclaurin polynomial of degree at least 4 to approximate $\int_0^{0.2} e^{-x^2}\,dx$.

I know $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!}$

So $e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!}$

$= 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}$

$\int_0^{0.2} e^{-x^2}\,dx = \int_0^{0.2} 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}\,dx$

$= \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42}\right]_0^{0.2} = 0.2 - \frac{(0.2)^3}{3} + \frac{(0.2)^5}{10} - \frac{(0.2)^7}{42}$