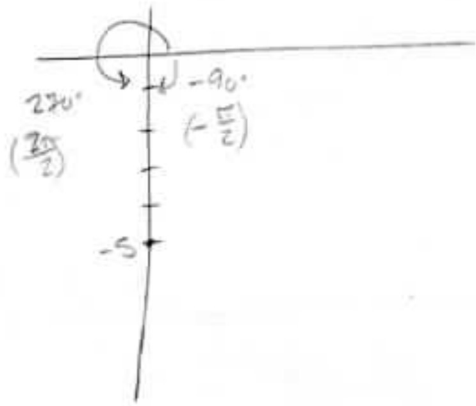
Exam 4 **Calculus 2** **4/21/22**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Convert the point with rectangular coordinates $(0, -5)$ to polar coordinates (r, θ).



$r = 5$

$\theta = \frac{3\pi}{2}$

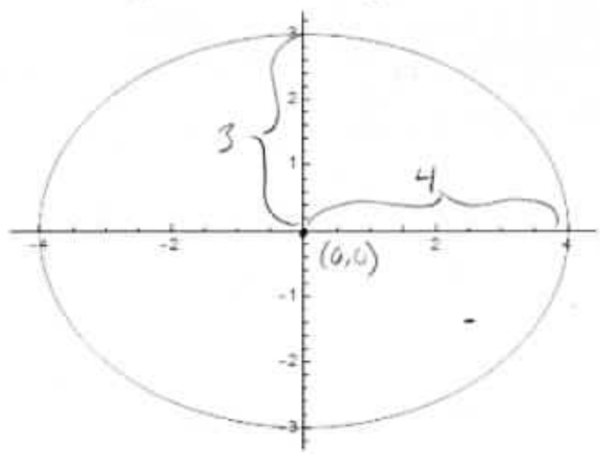
Great

$\left(5, \frac{3\pi}{2}\right)$

or

$\left(5, -\frac{\pi}{2}\right)$

2. Find an equation for the ellipse shown:



$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

Good

3. Find the slope of the tangent line to the polar curve $r = \sin(4\theta)$ at $\theta = \frac{\pi}{8}$.

$$\frac{\frac{d}{d\theta}(r\sin\theta)}{\frac{d}{d\theta}(r\cos\theta)} = \frac{\left[(\sin(4\theta))(\sin\theta)\right]'}{\left[(\sin(4\theta))(\cos\theta)\right]'}$$

$$= \frac{4\cos(4\theta)\sin\theta + \sin(4\theta)\cos\theta}{4\cos(4\theta)\cos\theta - \sin(4\theta)\sin\theta}$$

Excellent!

@ $\theta = \frac{\pi}{8}$

$$= \frac{4\cos\left(\cancel{4}\left(\frac{\pi}{\cancel{8}_2}\right)\right)\sin\left(\frac{\pi}{8}\right) + \sin\left(4\left(\frac{\pi}{\cancel{8}_2}\right)\right)\cos\left(\frac{\pi}{8}\right)}{4\cos\left(4\left(\frac{\pi}{\cancel{8}_2}\right)\right)\cos\left(\frac{\pi}{8}\right) - \sin\left(4\left(\frac{\pi}{\cancel{8}_2}\right)\right)\sin\left(\frac{\pi}{8}\right)}$$

$$= -2.414$$

4. Find the slope of the tangent line to the lemniscate with parametric equations $x(t) = 2\sin\frac{t}{2}$ and $y(t) = \sin t$ at the point $\left(1, \frac{\sqrt{3}}{2}\right)$.

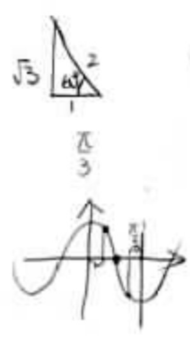
$$\frac{y'(t)}{x'(t)} = \frac{\cos(t)}{\cos(\frac{t}{2})}$$

$$1 = 2\sin\left(\frac{t}{2}\right) \Rightarrow \frac{1}{2} = \sin\left(\frac{t}{2}\right) \Rightarrow \frac{\pi}{6} = \frac{t}{2} \Rightarrow \frac{\pi}{3} = t$$

Nice

$$\frac{\cos\left(\frac{\pi}{3}\right)}{\cos\left(\frac{\pi}{6}\right)} = \frac{\sqrt{3}}{3} \approx 0.5774$$

5. Write an integral for the area of the region inside the inner loop of $r = 1 + 2\cos(\theta)$.



$$1 + 2\cos\theta = 0$$

$$2\cos\theta = -1$$

$$\cos\theta = -\frac{1}{2}$$

$$\theta = \frac{2}{3}\pi, \frac{4}{3}\pi$$

$$\int_{\frac{2}{3}\pi}^{\frac{4}{3}\pi} \frac{1}{2}\,(1 + 2\cos\theta)^2\, d\theta$$

Great!

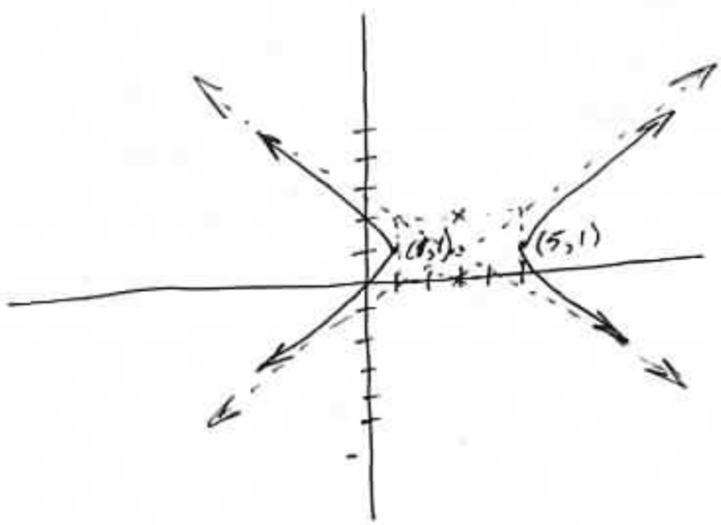
Hyperbola due to opposite signs on quadratic terms

6. Identify the graph of $\overbrace{x^2 - 4y^2}$ $- 6x + 8y + 1 = 0$ as a parabola, hyperbola, or ellipse, give coordinates of its vertices, and sketch a decent graph including any asymptotes.

$$x^2 - 6x + 9 \quad - 4(y^2 - 2y + 1) = -1 + 9 - 4$$

$$(x-3)^2 - 4(y-1)^2 = 4$$

$$\frac{(x-3)^2}{4} - \frac{(y-1)^2}{1} = 1$$



7. Bunny is a Calculus student at Enormous State University, and she's having some trouble. Bunny says "Ohmygod! Polar thingies are so pretty! But I'm totally geting stuck on the limits for integrating them, you know? I mean, 0 to 1 is always what I guess first, right, but that never seems to be right with these, so I guess 0 to 2 pi is like, normal or something? But sometimes it was just to pi or to 4 pi. How are you supposed to guess which?"

Help Bunny out by explaining how to determine the appropriate limits of integration for curves in polar coordinates.

So Bunny, first of all I definitely agree that polar graphs can really look amazing. But that goes with some tricky parts, and the limits of integration can really be among those tricky parts.

Those 0 to 1 limits that are the most typical back with regular integrals don't work nearly so often with these, mostly since trig functions have periods like 0 to 2π. So if you're looking for a default guess, that's probably it. But there are lots of exceptions, so think of it as a starting point. Try graphing it for θ from 0 to 2π. Sometimes that's conspicuously incomplete or unbalanced, and that's a clue. Regardless of that, it's a good idea to try more (like 0 to 4π) and less (like 0 to π) to see if you get more of the picture or keep exactly the same thing, which can mean your previous try was retracing at least part of the graph.

This is definitely not a time to feel bad about trial and error. Especially when there are negative r values involved, it can be too much to figure out reasonably without either technology or many hours.

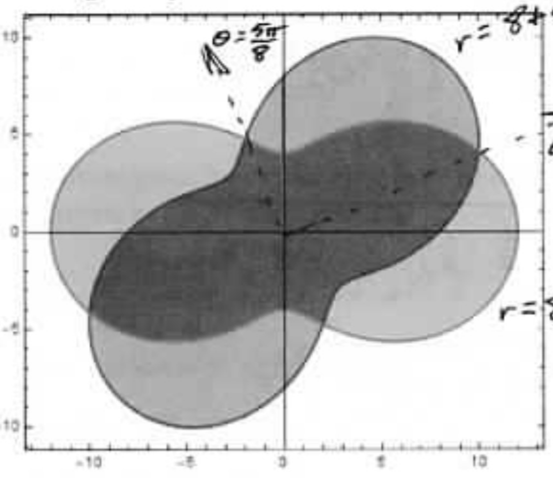
8. Write an integral for the length of the lemniscate with parametric equations $x(t) = 2\sin\frac{t}{2}$ and $y(t) = \sin t$.

$$\text{Length} = \int_{\alpha}^{\beta} \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt$$

$$= \int_0^{4\pi} \sqrt{\left(2\cos\frac{t}{2}\cdot\frac{1}{2}\right)^2 + (\cos t)^2}\, dt$$

$$= \int_0^{4\pi} \sqrt{\cos^2\frac{t}{2} + \cos^2 t}\, dt$$

9. Set up an integral for the area of the blue region in the plot below of $r = 8 + 4\cos 2\theta$ and $r = 8 + 4\sin 2\theta$, so the portion outside the horizontally oriented curve but inside the diagonally oriented curve.



When do they cross?

$$8 + 4\cos 2\theta = 8 + 4\sin 2\theta$$
$$4\cos 2\theta = 4\sin 2\theta$$
$$\cos 2\theta = \sin 2\theta$$
$$1 = \tan 2\theta$$
$$2\theta = \arctan 1$$
$$2\theta = \frac{\pi}{4} \text{ or } \frac{5\pi}{4}$$
$$\theta = \frac{\pi}{8} \text{ or } \frac{5\pi}{8}$$

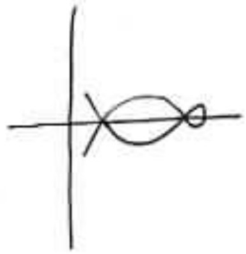
So the top blue region is:

$$\text{Area} = \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \frac{1}{2}\left(8 + 4\sin 2\theta\right)^2 d\theta - \int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \frac{1}{2}\left(8 + 4\cos 2\theta\right)^2 d\theta$$

Then the whole blue area is twice that, or:

$$\int_{\frac{\pi}{8}}^{\frac{5\pi}{8}} \left[\left(8 + 4\sin 2\theta\right)^2 - \left(8 + 4\cos 2\theta\right)^2\right] d\theta$$

10. Consider the curve with parametric equations $x(t) = 48 - 4t^2$, $y(t) = t^5 - 10t^3 + 9t$. Find the area of the entire region bounded by this curve.



When does it cross the x-axis?

$$0 = t^5 - 10t^3 + 9t$$

$$0 = t(t^4 - 10t^2 + 9)$$

$$0 = t(t^2 - 9)(t^2 - 1)$$

$$0 = t(t+3)(t-3)(t+1)(t-1)$$

$t = 0$ or $t = -3$ or $t = 3$ or $t = -1$ or $t = 1$

$$\int_{-3}^{-1} (t^5 - 10t^3 + 9t)\cdot(-8t)\,dt = \frac{5248}{7}$$

$$\int_{-1}^{0} \quad dt = \frac{-64}{7} \quad \text{(below x-axis)}$$

$$\int_{0}^{1} \quad dt = \frac{-64}{7} \quad \text{(right-to-left)}$$

$$\int_{1}^{3} \quad dt = \frac{5248}{7}$$

So the total area is $\frac{5248}{7} + \frac{64}{7} + \frac{64}{7} + \frac{5248}{7}$

$$= \frac{10624}{7} \approx 1517.\overline{714285}$$