Exam 2a **Calculus 2** **3/1/24**

Each problem is worth 10 points. For full credit provide good justification for your answers.

1. Set up an integral for the area of the region bounded between $y = x^2$ and $y = 2x$.

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$2x = x^2$

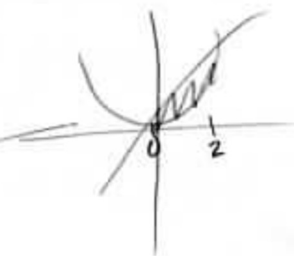
$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0 \quad x = 2$

$A = \int_0^2 (2x - x^2)\,dx$

Good



2. Set up an integral for the volume of the solid obtained when the region from #1 is rotated around the x-axis.

$V = \pi \int_a^b (f(x))^2\,dx$

$V = \int_0^2 \left[\pi (2x)^2 - \pi (x^2)^2\right] dx$

Good

3. If the work required to stretch a spring 1 ft beyond its natural length is 12 ft-lb, how much work (in ft-lb) is needed to stretch it 9 in. beyond its natural length?

$9 \text{ in} = .75$

$W = 12$

$12 = K\left[\frac{x^2}{2}\right]_0^1$

$12 = K\left(\frac{1}{2} - 0\right)$

$\underline{24 = K}$

$W = \int_0^{.75} 24x\,dx$

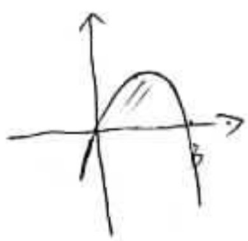
$\frac{24x^2}{2}\Big|_0^{.75}$

$12x^2\Big|_0^{.75}$

$12(.75)^2 - 0 = \underline{6.75 \text{ ft-lbs}}$

Excellent

4. Set up an integral for the length of the portion of the curve $y = x(3 - x)$ that lies above the x-axis.



$\text{length} = \underline{\int_0^3 \sqrt{1 + (-2x+3)^2}}$

Good

$y' = -2x + 3$

$y'^2 = (-2x+3)^2$

5. A bank account earns 6 percent interest compounded continuously. At what (constant, continuous) rate must a parent deposit money into such an account in order to save 100000 dollars in 18 years for a child's college expenses?

$$\text{Future Value} = \int_0^m P(t) \cdot e^{0.06(18-t)}\,dt$$

$$100000 = \int_0^{18} p \cdot e^{0.06(18-t)}\,dt$$

$$100000 = p \cdot \frac{-1}{0.06} \cdot e^{0.06(18-t)} \Big|_0^{18}$$

$$100000 = p \cdot \frac{-1}{0.06} \cdot \left(e^0 - e^{0.06 \cdot 18}\right)$$

$$100000 \approx p \cdot \frac{-1}{0.06} \cdot -1.94468$$

$$100000 \approx p \cdot 32.4$$

$$p \approx 3085.34$$

6. Find the x-coordinate of the center of mass of the region lying underneath the graph of the function $f(x) = \sqrt{x}$ over the interval $[0, 25]$.

$$\bar{x} = \frac{\int_0^{25} x \cdot \sqrt{x}\, dx}{\int_0^{25} \sqrt{x}\, dx}$$

$$= \frac{\int_0^{25} x^{\frac{3}{2}}\, dx}{\int_0^{25} x^{\frac{1}{2}}\, dx}$$

$$= \frac{\frac{2}{5} x^{\frac{5}{2}} \Big|_0^{25}}{\frac{2}{3} x^{\frac{3}{2}} \Big|_0^{25}}$$

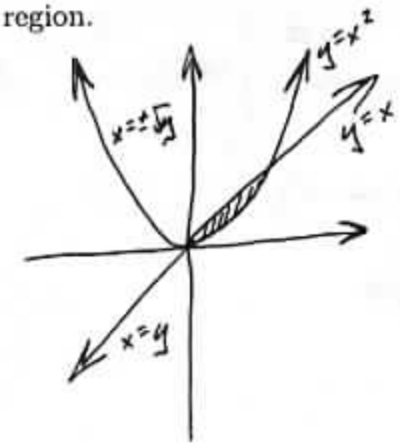
Great

$$= \frac{1250}{\frac{250}{3}}$$

$$= 1250 \times \frac{3}{250} = \underline{15}$$

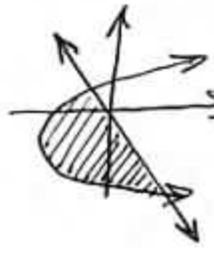
7. Star is a calculus student at Enormous State University, and they're having some trouble. Star says "Geez! Calc 2 is so different from Calc 1! It used to be there was, like, just one right way to do things, right? But now I did this area as $\int_0^1 (x - x^2)\, dx$ but the professor was saying you could do it $\int_0^1 (\sqrt{y} - y)\, dy$. Is it just a coincidence that they both work, or what?"

Help Star out. Explain to them as clearly as possible why both ways work for the same region.



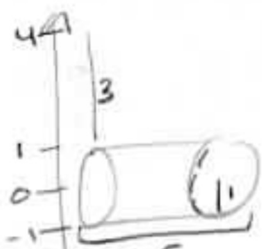
So Star, it totally makes sense that those work out the same, because they're two ways to do the exact same area. Look at this graph. The usual way to think about that shaded region is that it's under $y=x$ but above $y=x^2$. But you can also think about the same region as being left of $x=\sqrt{y}$ but right of $x=y$, so your professor's $\int_0^1 (\sqrt{y}-y)\,dy$ just does it sideways like that. It's between $x=0$ and $x=1$, but also between $y=0$ and $y=1$, so the limits work out the same.

Sometimes there's a region where only one way works well, like



works better if you do the "dy" way, but usually you get your pick.

8. A gas station stores its gasoline in a tank under the ground. The tank is a cylinder lying horizontally on its side. (In other words, the tank is not standing vertically on one of its flat ends.) If the radius of the cylinder is 1 meter, its length is 5 meters, and its top is 3 meters under the ground, set up an integral for the total amount of work needed to pump the gasoline out of the tank to ground level. (The density of gasoline is 673 kilograms per cubic meter; use $g = 9.8m/s^2$).



width: $2 \cdot \sqrt{1-x^2}$ m

area: $2 \cdot \sqrt{1-x^2} \cdot 5$ m^2

volume: $10\sqrt{1-x^2} \cdot \Delta x$ m^3

mass: $10\sqrt{1-x^2} \cdot 673 \frac{kg}{m^3} \cdot m^3 \cdot \Delta x$

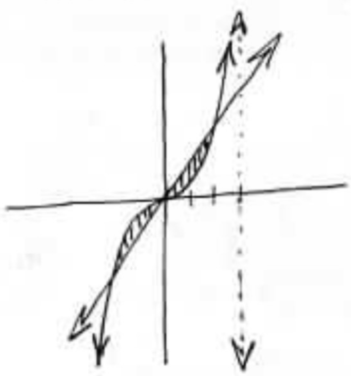
force: $6730\sqrt{1-x^2}$ kg $\cdot 9.8 m/s^2 \Delta x$

work= $65954\sqrt{1-x^2} \cdot (4-x)\Delta x$

total work = $\int_{-1}^{1} 65954\sqrt{1-x^2}(4-x)\,dx$

Excellent!

9. Consider the entire region bounded between $y = x^3$ and $y = 2x$. If this region is revolved around the axis $x = 3$, set up an integral or integrals for the volume of the resulting solid.

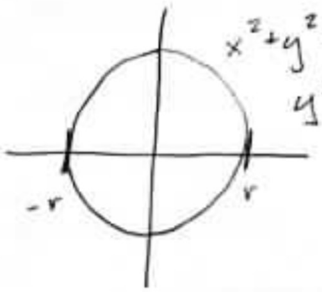


Shells

$$\int_0^{\sqrt{2}} 2\pi(3-x)(2x-x^3)\,dx$$

$$+\int_{-\sqrt{2}}^{0} 2\pi(3-x)(x^3-2x)\,dx$$

10. Consider a sphere with radius r. Treating it as a solid of revolution obtained from rotating half of a circle centered at the origin around one of the coordinate axes, show that the volume of the sphere is $\frac{4}{3}\pi r^3$.



$x^2+y^2=r^2$

$y=\sqrt{r^2-x^2}$

$$\text{Volume} = \int_{-r}^{r} \pi\left(\sqrt{r^2-x^2}\right)^2 dx$$

$$= \int_{-r}^{r} \pi\left(r^2-x^2\right)dx$$

$$= \pi \cdot \left(r^2 \cdot x - \frac{1}{3}x^3\right)\Big|_{-r}^{r}$$

$$= \pi \cdot \left[\left(r^3 - \frac{1}{3}r^3\right) - \left(-r^3 + \frac{1}{3}r^3\right)\right]$$

$$= \pi \cdot \left(\frac{2}{3}r^3 + \frac{2}{3}r^3\right)$$

$$= \frac{4}{3}\pi r^3$$