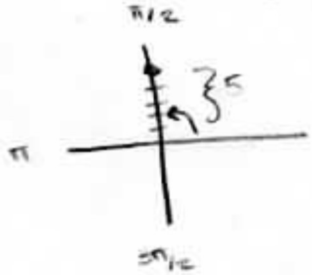
Exam 4a **Calculus 2** **4/19/24**

Each problem is worth 10 points. For full credit provide good justification for your answers.

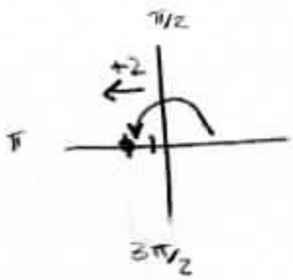
1. (a) Convert the point with rectangular coordinates $(0,5)$ to polar coordinates (r,θ).



$$(0,5) \Longrightarrow (r,\theta)$$
$$= (5, \pi/2)$$

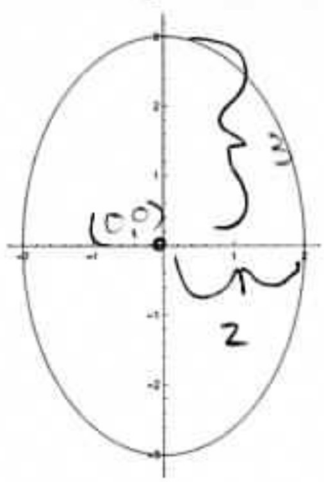
(b) Convert the point with polar coordinates $(2,\pi)$ to rectangular coordinates.

Good



$$(2,\pi) \Longrightarrow (-2,0)$$

2. Find an equation for the ellipse shown:



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

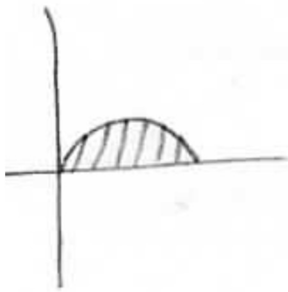
$$\frac{x^2}{4} + \frac{y^2}{9} = 1$$

Good

$x(t) = 3t - 3\sin t \qquad y(t) = 3 - 3\cos t$

$x'(t) = 3 - 3\cos t \qquad y'(t) = 3\sin t$

3. Write an integral for the length of one arch of the cycloid with parametric equations $x(t) = 3(t - \sin t)$, $y(t) = 3(1 - \cos t)$



$$\int_\alpha^\beta \sqrt{[x'(t)]^2 + [y'(t)]}$$

$\downarrow$

$$\int_0^{2\pi} \sqrt{(3-3\cos t)^2 + (3\sin t)^2}\, dt$$

where $y(t) = 0$

$3 - 3\cos t = 0$

$3 = 3\cos t$

$1 = \cos t$

$t = 0 \;\&\; 2\pi$

Great!

4. Write an integral for the area under one arch of the cycloid with parametric equations $x(t) = 3(t - \sin t)$, $y(t) = 3(1 - \cos t)$

area = $\int_\alpha^\beta y(t)\, x'(t)\, dt$

β & α same as before

$y t = 3(1 - \cos t)$

$x' t = 3 - 3\cos t$

Excellent!

$$A = \int_0^{2\pi} \underbrace{(3 - 3\cos t)}_{}^{yt} \overbrace{(3 - 3\cos t)}^{x't}\, dt$$

$$= \int_0^{2\pi} (3 - 3\cos t)^2\, dt$$

5. Write an integral for the area of the region inside the polar curve $r = \cos(5\theta)$.

$$10 \cdot \int_0^{\frac{\pi}{10}} \frac{1}{2}(\cos 5\theta)^2 \, d\theta$$

Good

 $\times 10 = A_{total}$

$r = 0 = \cos(5\theta)$

$\theta = \frac{1}{10}\pi$ $\quad \cos(5(\frac{1}{10}\pi)) = \cos(\frac{\pi}{2}) = 0$

6. Identify the graph of $4y^2 - 9x^2 + 36x = 72$ as a parabola, hyperbola, or ellipse, give coordinates of its vertices, and sketch a decent graph including any asymptotes.

$$4y^2 - 9x^2 + 36x = 72$$

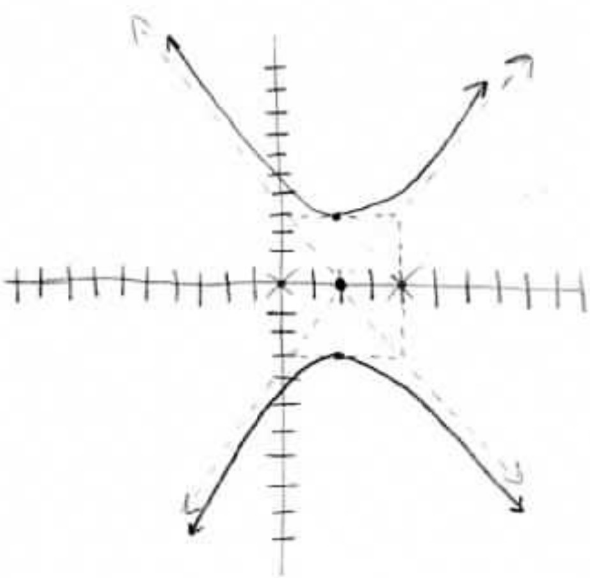
$$4y^2 - 9(x^2 - 4x + 4) = 72 - 36$$

$$\frac{4y^2}{36} - \frac{9(x-2)^2}{36} = \frac{36}{36}$$

$$\frac{y^2}{9} - \frac{(x-2)^2}{4} = 1$$

hyperbola
vertices: $(2,3)(2,-3)$

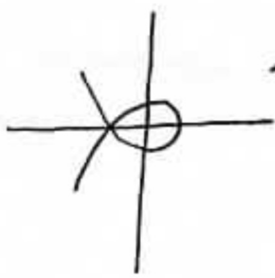
Excellent!



7. Star is a Calculus student at Enormous State University, and they're having some trouble. Star says "Geez, this parametric stuff is just totally confusing! Sometimes you set the x part equal to 0 to find limits, but sometimes you set y equal to 0 instead. How am I supposed to know which?"

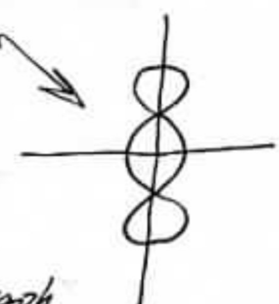
Help Star out by explaining when each approach might be appropriate.

Star, the most important thing is to let the graph help do part of the work for you. If a parametric graph looks like



this, then the loop starts and ends with points on the x-axis, right? So to find the t values for those points, set y(t) equal to 0 (yeah, it seems backwards that the x axis means y equals 0, but get used to it).

On the other hand if the graph were like



then the points that divide up the different parts are all on the y-axis, so those are where x is 0, right?

So the big thing is to look at a good graph, and then that tells you a lot about which axis helps you find limits.

8. Set up an integral or integrals for the area of the region between the loops of the curve with polar equation $r = 3 - 4\sin\theta$.

Where does it cross the origin?

$$0 = 3 - 4\sin\theta$$

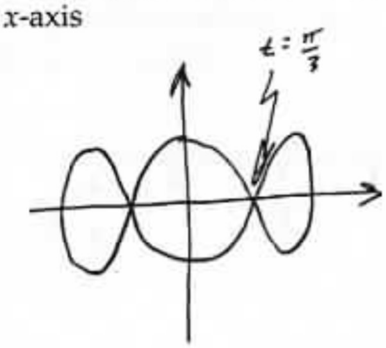
$$4\sin\theta = 3$$

$$\sin\theta = \frac{3}{4}$$

$$\left.\begin{aligned} \theta &= \sin^{-1}\tfrac{3}{4} \\ &\text{or} \\ \theta &= \pi - \sin^{-1}\tfrac{3}{4} \end{aligned}\right\}$$ These start and end the inner loop (from looking at graph)

$$\text{Area} = -\frac{1}{2}\int_{\sin^{-1}\frac{3}{4}}^{\pi - \sin^{-1}\frac{3}{4}} (3 - 4\sin\theta)^2\,d\theta + \int_{\pi - \sin^{-1}\frac{3}{4}}^{\sin^{-1}\frac{3}{4} + 2\pi} (3 - 4\sin\theta)^2\,d\theta$$

9. Consider the curve with parametric equations $x(t) = \cos t$, $y(t) = \sin 3t$. Find the slopes of the curve at each point where it crosses the x-axis



When does it cross the x-axis?

Put in 0 for y!

$0 = \sin 3t$

$3t = 0 + \pi \cdot k$ for any $k \in \mathbb{Z}$

$t = \frac{\pi}{3} \cdot k$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{3\cos 3t}{-\sin t}$$

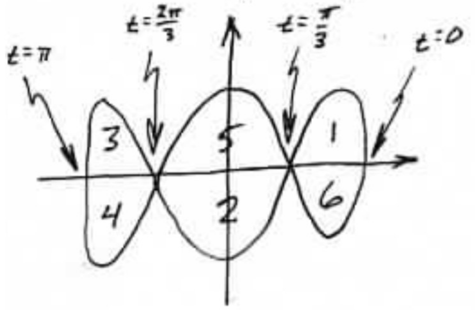
So when $t = \frac{\pi}{3}$: $\left.\frac{dy}{dx}\right|_{t=\frac{\pi}{3}} = \frac{3\cos\pi}{-\sin\frac{\pi}{3}} = \frac{3\cdot -1}{-\sqrt{3}/2} = 2\sqrt{3} \approx 3.4641$

when $t = \frac{2\pi}{3}$: $\left.\frac{dy}{dx}\right|_{t=\frac{2\pi}{3}} = \frac{3\cos 2\pi}{-2\sin\frac{2\pi}{3}} = \frac{3}{-\sqrt{3}/2} = -2\sqrt{3} \approx -3.4641$

when $t = \pi$: $\left.\frac{dy}{dx}\right|_{t=\pi} = \frac{3\cos 3\pi}{-\sin\pi} = \frac{3\cdot -1}{-\sin\pi} = \frac{-3}{0}$ undefined

Then it just repeats, so the slopes are $\pm 2\sqrt{3}$ or vertical

10. Consider the curve with parametric equations $x(t) = \cos t$, $y(t) = \sin 3t$. Set up one or more integrals to give the area inside the curve.



Set $y(t)=0$ to find limits:

$$0=\sin 3t$$
$$3t=\pi\cdot k \quad \text{for } k\in\mathbb{Z}$$
$$t=\frac{\pi}{3}\cdot k$$

So the graph gets traversed beginning at the rightmost point, enclosing regions in the order numbered in the figure.

Regions 1 and 3 are traversed right-to-left, so we'll get the negative of the area. Region 2 is backwards too, but also upside down, so it comes out right. If we do those and double, because of symmetry, then:

$$\text{Area} = 2\cdot\left[-\int_0^{\frac{\pi}{3}}(\sin 3t)(-\sin t)\,dt + \int_{\frac{\pi}{3}}^{\frac{2\pi}{3}}(\sin 3t)(-\sin t)\,dt - \int_{\frac{2\pi}{3}}^{\pi}(\sin 3t)(-\sin t)\,dt\right]$$