Examlet 1a **Foundations of Advanced Math** **2/2/24**

1. The square of a throddodd integer is throdd.

Let m be a throddodd integer, so $m = 3n+2$ for some $n \in \mathbb{Z}$.

Then
$$m^2 = (3n+2)^2 = 9n^2 + 12n + 4$$
$$= 9n^2 + 12n + 3 + 1$$
$$= 3(3n^2 + 4n + 1) + 1$$

But this means m^2 is throdd, since $3n^2+4n+1$ is an integer because of closure under multiplication and addition. □

2. $P \Rightarrow Q$ is logically equivalent to its contrapositive.

The contrapositive of $P \Rightarrow Q$ is $\neg Q \Rightarrow \neg P$

P	Q	$P \overset{*}{\Rightarrow} Q$	$\neg Q$	$\neg P$	$\neg Q \overset{*}{\Rightarrow} \neg P$
T	T	T	F	F	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

The truth values for ⊛ are same under possible circumstances. therefore $P \Rightarrow Q$ is logically equivalent to its contrapositive.

□

Excellent

3. If p, q, and r are integers for which $p|(q+r)$ and $p|q$, then $p|r$.

Well, if $p|(q+r)$ then $q+r = p \cdot a$ for some $a \in \mathbb{Z}$, and if $p|q$ then $q = p \cdot b$ for some $b \in \mathbb{Z}$.

But substituting $p \cdot b$ in for q in $q+r = p \cdot a$ we have

$$p \cdot b + r = p \cdot a$$

or

$$r = p \cdot a - p \cdot b$$

or

$$r = p(a-b).$$

Then since $a-b$ is also an integer by closure, this means $p|r$, as desired. $\square$

4. $\sqrt{2}$ is irrational.

Suppose $\sqrt{2}$ is rational. Then $\sqrt{2} = \frac{q}{r}$ for some $q, r \in \mathbb{Z}$ and are in its most simplest form (ex. $\frac{12}{5}$ cannot be furthered reducted). So if we square both sides; $2 = \frac{q^2}{r^2} \Rightarrow 2r^2 = q^2$. this implies that q^2 is divisible by 2 and based on our proofs, we are aware that 2 would also divide q. In doing this that would suggest q is an even number which has a definition of $q = 2m$ for some $m \in \mathbb{Z}$. Substituting for q; $2r^2 = (2m)^2 \Rightarrow 2r^2 = 4m^2$. We can then divide both sides by 2 granting $r^2 = 2m^2$. Now, we have seen a similar set up before in this statement. This indicates that r^2 is also divisible by two meaning r is divisible by two. However, we then have a contradiction where both q and r are divisible by two meaning they are not in their simplest form. Therefore, by contradiction $\sqrt{2}$ is irrational. $\square$

Excellent!

5. For any $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$ By induction

Prove Base case where $n=1$

$$\sum_{i=1}^{1} (1) = \frac{1(1+1)}{2} = \frac{(1)(2)}{2} = 1$$

The statement holds true for $n=1$. Assume this is true for ~~all~~ sort of. $n=K$, such that:

$$\sum_{i=1}^{K} i = \frac{K(K+1)}{2}$$

Now, it also needs to hold for $n=K+1$, so we will add $K+1$ to both sides

$$\sum_{i=1}^{K} i + K+1 = \frac{K(K+1)}{2} + K+1$$

[Common denominators

$$\sum_{i=1}^{K+1} i = \frac{K^2+K}{2} + \frac{2K+2}{2}$$

[add together

$$\sum_{i=1}^{K+1} i = \frac{K^2+3K+2}{2}$$

[factor

$$\sum_{i=1}^{K+1} i = \frac{(K+1)(K+2)}{2}$$

[manipulate

$$\sum_{i=1}^{K+1} i = \frac{K+1((K+1)+1)}{2}$$

This takes the same form $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ for $n=K+1$ thus true!

By induction, the statement holds true for any $n \in \mathbb{Z}^+$ Okay!